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Multisymplectic Manifolds and their Symmetries

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Abstract

Multisymplectic geometry is a branch of differential geometry concerned with manifolds equipped with a closed, non-degenerate differential form of degree $n \geq 2$. It generalizes symplectic geometry, which corresponds to the case $n = 2$ and provides the natural geometric framework for classical Hamiltonian mechanics. In the symplectic setting, the space of observables forms a Lie algebra, and the action of a Lie group is encoded by a Lie algebra morphism known as a (co)moment map. In higher-degree multisymplectic geometry, however, this familiar algebraic structure breaks down: the space of observables no longer forms a Lie algebra, and moment maps must be replaced by their higher analogues.

In this thesis, we introduce the framework of multisymplectic geometry and develop the algebraic and geometric tools necessary to study the space of observables and Hamiltonian Lie group actions on multisymplectic manifolds.

Contents

Introduction

Symplectic geometry, with its roots in classical mechanics, offers a mathematical structure to study the phase space in Hamiltonian dynamics. A symplectic manifold is a smooth manifold equipped with a closed, non-degenerate 2-form, called the symplectic form, which induces a natural correspondence between smooth functions and Hamiltonian vector fields. This correspondence encodes the dynamics on the manifold and underlies many fundamental constructions in differential geometry. The objective of this thesis is to study a generalization of these manifolds where the dynamics are induced by a higher-dimensional form, called multisymplectic geometry. Like symplectic manifolds describe the phase space in Hamiltonian mechanics, these multisymplectic manifolds describe multiphase spaces, which are crucial to the formulation of Hamiltonian classical field theories.

We begin in Chapter 1 by reviewing foundational material from symplectic geometry. We introduce the concept of a symplectic manifold, Hamiltonian vector fields, and the associated Poisson bracket on smooth functions. A key feature of symplectic geometry is that the space of smooth functions on a symplectic manifold, regarded as observables, naturally carries the structure of a Lie algebra, with the Poisson bracket satisfying the Jacobi identity and encoding the interaction of observables.

In the multisymplectic setting, however, this algebraic structure becomes more subtle. The generalization of the Poisson bracket to higher degree forms leads to operations that no longer satisfy the Jacobi identity strictly. Instead, one obtains a hierarchy of brackets that satisfy higher-order relations. This motivates the introduction of L_∞-algebras, a generalization of Lie algebras "up to homotopy". Chapter 2 is devoted to a detailed study of L_∞-algebras. We define them through the language of differential graded coalgebras and discuss the notion of L_∞-morphisms, which generalize Lie algebra homomorphisms in a way compatible with the higher structure.

Chapter 3 introduces multisymplectic manifolds, defined as manifolds equipped with a closed, nondegenerate $(n+1)$-form. We present several examples of such structures and discuss how key notions from symplectic geometry, such as Hamiltonian forms, observables, and their associated algebraic structures, admit natural generalizations. In particular, we show how the space of observables can be equipped with an L_∞-algebra structure [Rog12] that extends the Lie algebra structure familiar from the symplectic case.

Symmetry plays an important role in both the symplectic and multisymplectic settings. In symplectic geometry, one studies Lie group actions that preserve the symplectic form. If the infinitesimal generators of the action are Hamiltonian vector fields, and another compatibility condition is satisfied (the moment map being equivariant), it is called a Hamiltonian action, and is characterized by the existence of a moment map.

In Chapter 4, we investigate the extension of these ideas to the multisymplectic context. Here, the classical notion of a moment map is replaced by that of a homotopy moment map, defined as an L_∞-morphism from the Lie algebra of a group to the L_∞-algebra of observables on the multisymplectic manifold. A major focus of this chapter is the development of tools to determine when a given Lie group action on a multisymplectic manifold is Hamiltonian in this generalized sense. We formalize the necessary conditions for the existence of a homotopy moment map and work through explicit examples illustrating how these tools can be applied in practice. As part of the chapter, we present two small contributions: Alternate proofs of the following statements

- **Corollary 4.1.3**. In symplectic geometry, we know that if the symplectic form has a primitive that is invariant with respect to a Lie group action, then it admits a moment map. This statement also holds in the general multisymplectic case.
- **Proposition 4.2.6**. The Cartan model is one of the tools we use to prove the existence of a homotopy moment map. While defining it, we see that the multisymplectic form naturally appears as an element of the complex. In symplectic geometry, the existence of a moment map for a Lie group action can be interpreted as the presence of a 1-step extension of this form in the Cartan model. We show that an analogous statement holds in the multisymplectic case: the existence of a 1-step extension likewise guarantees the existence of a homotopy moment map.

Throughout the thesis, our approach is guided by the desire to generalize and unify structures from symplectic geometry within a higher-degree setting. In doing so, we aim to provide a self-contained and detailed account of how multisymplectic geometry, together with the theory of L_∞-algebras, forms a natural framework for understanding geometric structures that arise in the study of Hamiltonian formalisms of classical field theory.

Chapter 1

Symplectic Geometry

Symplectic geometry originates in classical mechanics: the phase space in a Hamiltonian system carries a natural, non-degenerate 2-form that encodes the laws of motion, conservation, and the interplay between coordinates. This geometric structure reformulates Hamilton's equations and makes the notion of observables and conserved quantities (via Poisson brackets) intrinsic. In this chapter, we cover all the prerequisites of symplectic geometry necessary for this thesis. Any reader interested in reading more is directed towards [DS01], which is our main reference for this section.

1.1 Basic Definitions

Definition 1.1.1. A **symplectic vector space** is a pair (V, Ω) where V is a real, finite-dimensional vector space and $\Omega \in \bigwedge^2 V^*$ is a non-degenerate form, i.e. its kernel:

$$\ker(\Omega) = \{v \in V : \Omega(v, w) = 0 \ \forall w \in V\}$$

is trivial.

Two symplectic vector spaces (V_1, Ω_1) and (V_2, Ω_2) are said to be symplectomorphic if there exists an isomorphism $F : V_1 \to V_2$ such that $F^*\Omega_2 = \Omega_1$.

Example 1.1.2. The most basic example of a symplectic vector space is $(\mathbb{R}^{2n}, \Omega_{can})$ where we consider $\mathbb{R}^{2n}$ with the standard basis $v_1, \ldots, v_n, w_1, \ldots, w_n$ and

$$\Omega_{can}(v_i, v_j) = 0, \quad \Omega_{can}(w_i, w_j) = 0, \quad \Omega_{can}(v_i, w_j) = \delta_{ij}$$

It can be proved that every symplectic vector space is symplectomorphic to $(\mathbb{R}^{2n}, \Omega_{can})$ for some $n \in \mathbb{N}$.

Definition 1.1.3. Let M be a manifold. $\omega \in \Omega^2(M)$ is said to be a **symplectic form** if it is closed and ω_p is non-degenerate $\forall p \in M$ i.e.,

$$ker(\omega_p) := \{v \in T_pM | \ \omega_p(v, w) = 0 \ \forall w \in T_pM\}$$

is trivial $\forall p \in M$.

The pair (M, ω) is called a 'symplectic manifold'.
We now define the notion of isomorphism between two symplectic manifolds.

Definition 1.1.4. A **symplectomorphism** between two symplectic manifolds (M_1, ω_1) and (M_2, ω_2) is a diffeomorphism $\phi : M_1 \to M_2$ such that $\phi^*(\omega_2) = \omega_1$.

Example 1.1.5. Consider $M = \mathbb{R}^{2n}$ with the coordinates $(e_1, \dots, e_n, f_1, \dots, f_n)$. Then $(\mathbb{R}^{2n}, \sum_{i=1}^n e_i \wedge f_i)$ is a symplectic manifold.

Example 1.1.6. If M is a surface and $\omega \in \Omega^2(M)$ is a volume form on M, then (M, ω) is a symplectic manifold.

Example 1.1.7. Let Q be a manifold and consider $M = T^*Q$, i.e., the cotangent bundle of Q. Such a manifold is always equipped with a tautological form $\theta \in \Omega^2(M)$ defined by

$$\theta_{\eta_y}(v) = \eta_y(\pi_* v)$$

where $y \in Q$, $\eta_y \in T_y^*Q$, $v \in T_{\eta_y}(T^*Q)$ and $\pi : T^*Q \to Q$.
Then, $(M, -d\theta)$ is a symplectic manifold. We express θ in local coordinates. Let $\bar{y_1}, \dots, \bar{y_n} \in C^\infty(U)$ where $U \subset Q$ be a system of local coordinates. Then they induce a system of coordinates $y_1, \dots y_n, p^1, \dots, p^n \in C^\infty(T^*Q|_U)$, where $y_i = \bar{y_i} \circ \pi$ and

$$\begin{aligned} p^i : T^*Q|_U &\longrightarrow \mathbb{R} \\ \zeta &\mapsto \zeta\left(\frac{\partial}{\partial y_i}\right) \end{aligned}$$

Consider $\zeta = \sum_i a^i (d\bar{y_i})_x \in T_x^*Q$ and $v \in T_\zeta(T^*Q)$. Here, note that $p^i(\zeta) = a^i$ and $\pi^* d\bar{y_i} = dy_i$. So, we have

$$\begin{aligned} \theta_\zeta(v) &= \zeta(\pi_* v) \\ &= \sum_i a^i (d\bar{y_i})_x (\pi_* v) \\ &= \sum_i a^i (\pi^* d\bar{y_i})_\zeta (v) \\ &= \sum_i p^i(\zeta)(\pi^* d\bar{y_i})_\zeta (v) \\ &= \sum_i (p^i dy_i)_\zeta (v) \end{aligned}$$

Thus, in local coordinates, we have $\theta = \sum_i p^i dy_i$. So

$$\omega = \sum_i dy_i \wedge dp^i$$

Note that we get the symplectic case for $n = 1$.

From here on, unless mentioned otherwise, assume (M, ω) to be a symplectic manifold. We now define the concept of a Hamiltonian vector field.

Definition 1.1.8. Given $f \in C^\infty(M)$, there exists a unique vector field $X_f \in \mathfrak{X}(M)$ such that $\iota_X \omega = -df$. Such a vector field is called as the **Hamiltonian vector field** of f.

Note that all Hamiltonian vector fields are symmetries of the symplectic form i.e. $\mathcal{L}_{X_f}\omega = 0$ for all $f \in C^\infty(M)$. We call all the vector fields $X \in \mathfrak{X}(M)$ which are symmetries of ω i.e., which obey $\mathcal{L}_X\omega = 0$ as *symplectic* vector fields. We denote by $\mathfrak{X}_{Ham}(M)$ and $\mathfrak{X}_{Sym}(M)$ the set of all Hamiltonian and symplectic vector fields of M.
The reason each function is associated with a unique Hamiltonian vector field is because the symplectic form ω is non-degenerate, thus it induces an isomorphism

$$\begin{aligned}\tilde{\omega} : \mathfrak{X}(M) &\to \Omega^1(M)\\ X &\mapsto \iota_X\omega\end{aligned}$$

Note that $\tilde{\omega}(\mathfrak{X}_{Ham}(M)) = Z^1(M)$, i.e. the space of exact 1-forms and $\tilde{\omega}(\mathfrak{X}_{Sym}(M)) = B^1(M)$, i.e. the space of closed 1-forms. Combining these results, we have the following:

Proposition 1.1.9. *Given a symplectic manifold (M, ω), $\mathfrak{X}_{Ham} \subseteq \mathfrak{X}_{Sym}$, and*

$$H^1_{dR}(M) = \frac{\mathfrak{X}_{Sym}(M)}{\mathfrak{X}_{Ham}(M)}$$

If we have two symplectic vector fields $X_1, X_2 \in \mathfrak{X}_{Sym}(M)$, we have that $[X_1, X_2]$ is a Hamiltonian vector field corresponding to the function $\omega(X_1, X_2)$.
Thus, $\mathfrak{X}_{Sym}(M)$ and $\mathfrak{X}_{Ham}(M)$ are Lie subalgebras of $\mathfrak{X}(M)$.
We also have the following exact sequence

$$0 \longrightarrow \mathbb{R} \hookrightarrow C^\infty(M) \longrightarrow \mathfrak{X}_{Ham}(M) \longrightarrow 0$$

A consequence of this is that $C^\infty(M)$ can then be endowed with the Lie algebra structure, which turns the above into a short exact sequence of Lie algebras.

Definition 1.1.10. The **Poisson bracket** of $f, g \in C^\infty(M)$ is defined as

$$\{f, g\} := \omega(X_f, X_g) = df(X_g) = \mathcal{L}_{X_g}(f)$$

Lemma 1.1.11. *The Poisson bracket follows the Jacobi identity, i.e.*

$$[[\alpha_1, \alpha_2], \alpha_3] - [[\alpha_1, \alpha_3], \alpha_2] - [\alpha_1, [\alpha_2, \alpha_3]] = 0.$$

Proof. Let $\alpha_1, \alpha_2, \alpha_3 \in \Omega^1_{Ham}(M)$ and $X_1, X_2, X_3 \in \mathfrak{X}_{Ham}(M)$ be their respective Hamiltonian vector fields. Using the fact that $\mathcal{L}_{X_i}\omega = 0$ for $i = 1, 2, 3$ and ω is closed, Cartan's magic formula: $\mathcal{L}_X\omega = d\iota_X\omega + \iota_X d\omega$ and the identity $\iota_{[X,Y]} = \mathcal{L}_X\iota_Y - \iota_X\mathcal{L}_Y$, we get that

$$\begin{aligned}
0 &= \iota_{X_3}\iota_{X_2}\iota_{X_1}d\omega\\
&= \iota_{X_3}\iota_{X_2}(\mathcal{L}_{X_1}\omega - d\iota_{X_1}\omega)\\
&= -\iota_{X_3}\iota_{X_2}d\iota_{X_1}\omega\\
&= -\iota_{X_3}(\mathcal{L}_{X_2}\iota_{X_1}\omega - d\iota_{X_2}\iota_{X_1}\omega)\\
&= -\iota_{X_3}(\iota_{[X_2,X_1]}\omega + \iota_{X_1}\mathcal{L}_{X_2}\omega - d\iota_{X_2}\iota_{X_1}\omega)\\
&= \iota_{X_3}d\iota_{X_2}\iota_{X_1}\omega - \iota_{X_3}\iota_{[X_2,X_1]}\omega\\
&= \mathcal{L}_{X_3}\iota_{X_2}\iota_{X_1}\omega - d\iota_{X_3}\iota_{X_2}\iota_{X_1}\omega + \iota_{X_3}\iota_{[X_1,X_2]}\omega\\
&= \iota_{[X_3,X_2]}\iota_{X_1}\omega + \iota_{X_2}\mathcal{L}_{X_3}\iota_{X_1}\omega - d\iota_{X_3}\iota_{X_2}\iota_{X_1}\omega + \iota_{X_3}\iota_{[X_1,X_2]}\omega\\
&= \iota_{X_2}\iota_{[X_3,X_1]}\omega + \iota_{X_2}\iota_{X_1}\mathcal{L}_{X_3}\omega - \iota_{[X_2,X_3]}\iota_{X_1}\omega - d\iota_{X_3}\iota_{X_2}\iota_{X_1}\omega + \iota_{X_3}\iota_{[X_1,X_2]}\omega\\
&= \iota_{X_3}\iota_{[X_1,X_2]}\omega - \iota_{X_2}\iota_{[X_1,X_3]}\omega - \iota_{[X_2,X_3]}\iota_{X_1}\omega - d\iota_{X_3}\iota_{X_2}\iota_{X_1}\omega\\
&= [[\alpha_1, \alpha_2], \alpha_3] - [[\alpha_1, \alpha_3], \alpha_2] - [\alpha_1, [\alpha_2, \alpha_3]]
\end{aligned}$$

In the final equality, we used the fact that $d\iota_{X_3}\iota_{X_2}\iota_{X_1}\omega = 0$ as ω is a 2-form. □

Thus, we note that $(C^\infty(M), \{\cdot,\cdot\})$ is a Lie algebra. It is referred to as the 'Lie algebra of observables'. Note that $\{\cdot,\cdot\}$ is a biderivation of the product, i.e., given $f, g, h \in C^\infty(M)$, we have that

$$\{f.g, h\} = \{f, h\}.g + f.\{g, h\}$$

Thus, the space of observables is in fact a Poisson algebra.

1.2 Hamiltonian actions

Consider the action of a Lie group $G \curvearrowright M$. We assume that G acts on M by symplectomorphisms, i.e. the diffeomorphism $\Phi_g : M \to M : p \mapsto g \cdot p$ for all $g \in G$ is a symplectomorphism.

Given the action of a Lie group, we have its associated infinitesimal action

$$\phi : \mathfrak{g} \to \mathfrak{X}(M) : v \mapsto \tilde{v}, \quad \tilde{v}_p = \left(\frac{d}{dt}\right)_{t=0} \Phi(exp(-tv), p)$$

where $\mathfrak{g}$ is the Lie algebra associated to G.
If G acts on M by symplectomorphisms, the associated infinitesimal action will be via symplectic vector fields i.e. $\mathcal{L}_{\tilde{v}}\omega = 0$ for all $v \in \mathfrak{g}$. Moreover, if $\tilde{v}$ is a Hamiltonian vector field for all $v \in \mathfrak{g}$, the action of G on M is said to be *Hamiltonian*.
We describe Hamiltonian actions of Lie groups via a 'moment map'.

Definition 1.2.1. Assume that G acts on M by symplectomorphisms. Then, a **moment map** for the action is a smooth map

$$\mu : M \to \mathfrak{g}^*$$

satisfying

1. $d\mu^v = -\iota_{\tilde{v}}\omega$ for all $v \in \mathfrak{g}$,
   where $\mu^v \in C^\infty(M)$ such that $\mu^v(p) := \langle \mu(p), v \rangle$

2. μ is G-equivariant i.e.

$$\forall g \in G \quad \mu(g \cdot p) = (Ad^*)_g(\mu(p))$$

   where Ad^* is coadjoint action of G on $\mathfrak{g}^*$

The action is said to be Hamiltonian if there exists a moment map.

For connected Lie groups, a Hamiltonian action can be equivalently described in terms of a comoment map

$$\mu^* : \mathfrak{g} \to C^\infty(M)$$

which satisfies the following conditions:

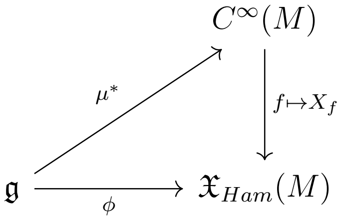
1. $\mu^*(v)$ is the Hamiltonian function corresponding to $\tilde{v}$ for all $v \in \mathfrak{g}$

2. μ^* is a Lie algebra morphism i.e.

$$\mu^*([v,w]) = \{\mu^*(v), \mu^*(w)\}$$

where $\{\cdot,\cdot\}$ is the Poisson bracket on M

Thus the map μ^* makes the following diagram commutative:



Example 1.2.2. Consider the action of the Lie group $(\mathbb{R}^3, +)$ on $(\mathbb{R}^3 \times \mathbb{R}^3, \omega = \sum_i dq_i \wedge dp_i)$ by translations, i.e.

$$(x_1, x_2, x_3).((q_1, q_2, q_3), (p_1, p_2, p_3)) = ((q_1 + x_1, q_2 + x_2, q_3 + x_3), (p_1, p_2, p_3))$$

The corresponding Lie algebra $(\mathbb{R}^3, \times)$ (where $\times$ is the vector cross product) is isomorphic to $\mathfrak{so}(3)$. Its infinitesimal action is given by

$$v_{(v_1,v_2,v_3)} = -\sum_i v_i \frac{\partial}{\partial q_i}$$

Note that $\iota_{v_{(v_1,v_2,v_3)}}\omega = -\sum_i v_i dp_i = -d(\sum_i v_i p_i)$, i.e. the action is Hamiltonian. Using $(\mathbb{R}^3)^* = \mathbb{R}^3$, the moment map for this action can be written as

$$J : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3, \qquad J((q_1, q_2, q_3), (p_1, p_2, p_3)) = (p_1, p_2, p_3)$$

Example 1.2.3. Consider $\mathbb{R}^3 \setminus \{(0,0,x) : x \in \mathbb{R}\}$ with cylindrical coordinates r, θ, z. The unit sphere S^2 is the set $r^2 + z^2 = 1$. The 2-form $d\theta \wedge dz$, which is defined away from the z-axis, restricts to the standard symplectic form on the unit sphere.

Let us consider the action of the group $SO(2)$ on the unit sphere by rotations about the z-axis:

$$\Psi = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SO(2)$$

The matrix representation given above is for $\mathbb{R}^3$ given in cartesian coordinates. In cylindrical coordinates, the action is given as

$$\Psi \cdot (r, \theta, z) := (r, \theta + \psi, z)$$

for $(r, \theta, z) \in \mathbb{R}^3$. The Lie algebra of $SO(2)$ is $\mathbb{R}$ and the infinitesimal action is:

$$\rho : \mathbb{R} \to \mathfrak{X}(\mathbb{R}^3) : v \mapsto -v\frac{\partial}{\partial\theta}.$$

Since $\iota_{\frac{\partial}{\partial\theta}}\omega = -dz$, we see that the "height function"

$$J : S^2 \to \mathbb{R}^* \cong \mathbb{R}$$
$$(\theta, z) \mapsto z$$

is a moment map for this action.

Example 1.2.4 (Symplectic forms with invariant primitives)**.** Consider a symplectic action of the Lie group G on the symplectic manifold $(M, \omega = d\alpha)$. If the primitive α is G-invariant, then the action is Hamiltonian, and there exists a moment map given by

$$J : M \to \mathfrak{g}^* \quad \langle J(p), x\rangle = (\iota_{v_x}\alpha)(p)$$

To show that J is a moment map, we check requirement (i) in Definition 1.1.9:

$$\iota_{v_x}\omega = \iota_{v_x} d\alpha = -d\iota_{v_x}\alpha + \mathcal{L}_{v_x}\alpha = -d\mu(x).$$

Now we need to show that J is equivariant. Given $g \in G$, let $\varphi_g : M \to M$ be the associated diffeomorphism. Then

$$\begin{aligned}
\langle J(g \cdot p), x\rangle &= (\iota_{v_x}\alpha)(g \cdot p) \\
&= \left[\varphi_g^*(\iota_{v_x}\alpha)\right](p) \\
&= \left[\iota_{(\varphi_{g^{-1}})_* v_x}\varphi_g^*\alpha\right](p) \\
&= \left[\iota_{(\varphi_{g^{-1}})_* v_x}\alpha\right](p) \\
&= \left[\iota_{\rho(\mathrm{Ad}_{g^{-1}}x)}\alpha\right](p) \\
&= \langle J(p), \mathrm{Ad}_{g^{-1}}x\rangle \\
&= \langle(\mathrm{Ad}_g^* J(p)), x\rangle.
\end{aligned}$$

In the third last equality, we use that

$$g_*(v_x) = v_{Ad_g x}$$

This in turn is a consequence of the fact that for any $g \in G, x \in \mathfrak{g}$, we have that

$$g \cdot exp(x) \cdot g^{-1} = exp(Ad_g x)$$

We end this section with some results regarding the existence and uniqueness of moment maps.

Proposition 1.2.5 (Proposition 26.5 in [DS01])**.** *Consider a Hamiltonian Lie group action $G \curvearrowright (M, \omega)$. If $H^1(\mathfrak{g}) = 0$, then the moment maps for the action are unique.*

Note: For details regarding the definition of the *Chevalley-Eilenberg* cohomology $H^\bullet(\mathfrak{g})$, refer to section 4.1.

Proof. Let $\mu_1, \mu_2 : \mathfrak{g} \to C^\infty(M)$ be two moment maps corresponding to the G-action. Thus, $\mu_1(x)$ and $\mu_2(x)$ are the Hamiltonian functions corresponding to the same Hamiltonian vector field v_x, which implies that $(\mu_1 - \mu_2)(x)$ is a constant function. We denote it as $c : \mathfrak{g} \to \mathbb{R} : x \mapsto \mu_1(x) - \mu_2(x)$. As μ_1 and μ_2 only differ by a constant, we can say that for all $x, y \in \mathfrak{g}$,

$$\begin{aligned}
&\{\mu_1(x), \mu_1(y)\} = \{\mu_2(x), \mu_2(y)\} \\
\implies &\mu_1([x, y]) = \mu_2([x, y])
\end{aligned}$$

Thus, $c \in [\mathfrak{g}, \mathfrak{g}]^0 = \{0\} \implies \mu_1 = \mu_2$. □

Building off this proposition, we have the following result:

Proposition 1.2.6 (Proposition 26.3 in [DS01]). *Consider a symplectic action of a Lie group $G \curvearrowright (M, \omega)$ such that $H^1(\mathfrak{g}) = H^2(\mathfrak{g}) = 0$. Then there exists a unique moment map for the action.*

Here, we remark that semi-simple Lie algebras satisfy $H^1(\mathfrak{g}) = H^2(\mathfrak{g}) = 0$. Thus, any symplectic action of a semi-simple Lie group is Hamiltonian.

Chapter 2

L_∞ - algebras

On a symplectic manifold, the space of smooth functions naturally acquires the structure of a Lie algebra via the Poisson bracket, which satisfies bilinearity, antisymmetry, and the Jacobi identity. This Lie algebra is often referred to as the Lie algebra of observables, as it encodes the algebraic structure of classical physical quantities. However, in the context of multisymplectic manifolds, it is no longer possible to define a binary bracket that satisfies the Jacobi identity in the traditional sense.

To address this, one can instead generalize the notion of a Lie algebra itself. This leads to the concept of L_∞-algebras, which are graded vector spaces equipped with a collection of multilinear maps satisfying a series of generalized Jacobi identities. As such, L_∞-algebras provide a natural framework for describing observables in multisymplectic geometry and other contexts where classical Lie algebra structures are insufficient.

The goal of this chapter is to introduce the formalism of L_∞-algebras and lay the groundwork for their use in later chapters. In particular, we begin by defining L_∞-algebras and exploring their basic properties. We then turn to their underlying differential graded coalgebra structure, which provides a compact and elegant way of encoding the structure corresponding to the generalized Jacobi identities. This perspective is particularly useful in the next section, where we define L_∞-morphisms, which serve as the appropriate notion of structure-preserving maps between L_∞-algebras and are essential for applications such as the definition of homotopy moment maps.

2.1 Graded vector spaces and coalgebras

This section aims to introduce relevant background that is essential for the rest of this thesis. We shall mainly refer to [Sha14] for this section.

Graded vector spaces

Definition 2.1.1. A **graded vector space** over $\mathbb{R}$ is a real vector space V, together with a family of sub-spaces $(V_i)_{i\in\mathbb{Z}}$, such that $V = \bigoplus_{i\in\mathbb{Z}} V_i$.

The homogeneous elements of degree $i \in \mathbb{Z}$ of a graded vector space $V = \bigoplus_{i\in\mathbb{Z}} V_i$ are the elements of V_i. Given a graded vector space V, then sV denotes the suspension and $s^{-1}V$ denotes the desuspension of V, defined respectively by

$$(sV)_i = V_{i-1}, \qquad (s^{-1}V)_i = V_{i+1}$$

Another useful notation for the suspension (and desuspension) of a graded vector space V is

$$s^{\pm k}V = V[\mp k]$$

We will use the two notations interchangeably.

Definition 2.1.2. A **morphism** f **of degree k** ($k \in \mathbb{Z}$) from a $\mathbb{Z}$-graded Vector Space V to a $\mathbb{Z}$-graded vector space W is a collection of maps $(f_i : V_i \to W_{i+k})_{i\in\mathbb{Z}}$

The degree of a morphism f is denoted by $|f|$.

Definition 2.1.3. A **real graded algebra** is a real graded vector space V equipped with a bilinear product $V \times V \to V$ denoted by $\cdot$ called multiplication (or concatenation of elements), such that

$$V_i \cdot V_j \subset V_{i+j}$$

Definition 2.1.4. Let V be a graded vector space. The **tensor algebra** $\mathcal{T}(V)$ is the graded vector space given by the collection of vector spaces:

$$\mathcal{T}(V)_m = \bigoplus_{k\geq 0} \bigoplus_{j_1+\dots+j_k=m} V_{j_1} \otimes \dots \otimes V_{j_k}, \qquad m \in \mathbb{Z}$$

Every component $\mathcal{T}(V)_m$ can be decomposed to the tensor product degree $\otimes$ as follows

$$\mathcal{T}^k(V)_m := \mathcal{T}(V)_m \cap V^{\otimes k}, \ \ k \geq 1$$

For $k = 0$, we set $\mathcal{T}^0(V) = \mathbb{R}$.

The vector space $\mathcal{T}^k(V)_m$ carries two natural actions, *even* and *odd*, of the group S_k of the permutations of k elements.
Given an arbitrary element $\sigma \in S_k$, the even representation is defined as

$$\sigma \cdot (x_1 \otimes \dots \otimes x_k) = \epsilon(\sigma; x_1, \dots, x_k)\ x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(k)}$$

and the odd representation is defined as

$$\sigma \cdot (x_1 \otimes \dots \otimes x_k) = (-1)^\sigma \epsilon(\sigma; x_1, \dots, x_k)\ x_{\sigma(1)} \otimes \dots \otimes x_{\sigma(k)}$$

Here, given $x_1, \dots, x_k \in V$, $\epsilon(\sigma; x_1, \dots, x_k)$ is the *Koszul sign* which is defined by the equality

$$x_1 \cdots x_n = \epsilon(\sigma; x_1, \dots, x_n)\ x_{\sigma(1)} \cdots x_{\sigma(k)}$$

which holds in the free graded commutative algebra generated by V.

Definition 2.1.5. Given a graded vector space V, the **graded symmetric algebra** $\mathcal{S}(V)$ is the graded vector space whose elements are the invariants of the even representation of S on $\mathcal{T}(V)$ with inherited grading.

Definition 2.1.6. Given a graded vector space V, the **graded antisymmetric algebra** $\Lambda(V)$ is the graded vector space whose elements are the invariants of the odd representation of S on $\mathcal{T}(V)$ with inherited grading.

The vector spaces defined above are, in a sense, isomorphic to each other. The isomorphism, known as the décalage isomorphism, is defined for $k \in \mathbb{N}$, as:

$$dec_k : \Lambda^k(V)[k] \to S^k(V[1])$$
$$x_1 \cdots x_k \mapsto (-1)^{(k-1)|x_1|+\dots+2|x_{k-1}|} x_1 \cdots x_k$$

Coalgebras

Coalgebras are structures that, in a category-theoretic sense, are dual to algebras. An algebra $(V, \cdot)$ is a vector space V equipped with a product, defined as follows:

$$V \otimes V \to V$$
$$(x_1, x_2) \mapsto x_1 \cdot x_2$$

An algebra is said to be associative if the following diagram is commutative:

$$\begin{CD}
V \otimes V \otimes V @>{\cdot \otimes id}>> V \otimes V \\
@V{id \otimes \cdot}VV @VV{\cdot}V \\
V \otimes V @>>{\cdot}> V
\end{CD}$$

Reversing the arrows in the above definition gives us the following

Definition 2.1.7. A **graded coalgebra** (C, Δ) is a graded vector space C equipped with a linear map $\Delta : C \to C \otimes C$ called co-multiplication, such that

$$\Delta(C_i) \subset \bigoplus_{j+k=i} C_j \otimes C_k$$

A coalgebra is said to be coassociative if the following diagram commutes:

$$\begin{CD}
C @>{\Delta}>> C \otimes C \\
@V{\Delta}VV @VV{\Delta \otimes id}V \\
C \otimes C @>>{id \otimes \Delta}> C \otimes C \otimes C
\end{CD}$$

Similarly, we can define cocommutativity by considering the dual condition of commutativity in an algebra. Given a coalgebra (C, Δ), consider $T : C \otimes C \to C$, the Twist map defined as $T(x \otimes y) = (-1)^{|x||y|}\ x \otimes y$ on homogeneous elements. A coalgebra (C, Δ) is said to be cocommutative iff $T \circ \Delta = \Delta$.
Given a graded vector space V, its symmetric algebra $\mathcal{S}$ expressed as

$$\mathcal{S}(V) = \bigoplus_{k=0}^{\infty} S^k(V) = \mathbb{R} \oplus \bar{S}(V)$$

is a cocommutative coalgebra in a natural way. Here,

$$S^0(V) := \mathbb{R} \qquad \text{and} \qquad \bar{S}(V) := \bigoplus_{k=1}^{\infty} S^k(V)$$

There is an induced coalgebra structure on $\bar{S}(V)$ along with an induced deconcatenation product $\bar{\Delta}$ called *reduced comultiplication*.

Given an algebra $(V,\ \cdot)$, we know that a derivation is a linear map $D : V \to V$ which follows the Leibniz identity, i.e. for all $v, w \in V$

$$D(v \cdot w) = D(v) \cdot w + v \cdot D(w)$$

We now define the dual notion of the derivation for a graded coalgebra.

Definition 2.1.8. A **coderivation** of degree one on a coalgebra (C, Δ) is a linear map $Q : C^i \to C^{i+1}$ satisfying $Q \circ Q = 0$ and the co-Leibniz identity

$$\Delta Q = (Q \otimes id)\Delta + (id \otimes Q)\Delta$$

Now, for the final result in this section, we try to describe a coderivation via a collection of morphisms. We know that a derivation on an algebra is uniquely determined by how it acts on the generators. A similar result can be given for coderivations of coalgebras.

We consider a specific case $C = \bar{S}(V)$, where V is a graded vector space. We define the following restrictions and projections:

$$Q_m = Q|_{\bar{S}^m(V)} : \bar{S}^m(V) \to \bar{S}(V),\ \ 1 \le m < \infty$$

so that $Q = \sum_{k=1}^{\infty} Q_k$. Also,

$$Q_m^k = pr_{\bar{S}^k(V)} \circ Q|_{\bar{S}^m(V)} : \bar{S}^m(V) \to \bar{S}^k(V)$$

Definition 2.1.9. $\sigma \in S_{p+q}$ is said to be a **(p,q)-unshuffle** if for $i \neq p, \sigma(i) < \sigma(i+1)$. The set of all (p,q)-unshuffles is denoted by Sh(p,q).

For example, we have that $e,\ (1\ 2),\ (1\ 2\ 3) \in \mathsf{Sh}(2,1)$ and $e,\ (2\ 3),\ (1\ 2\ 4\ 3) \in \mathsf{Sh}(2,2)$.

Proposition 2.1.10. *A coderivation Q of $\bar{S}(V)$ is uniquely determined by the collection $Q_i^1,\ i \in \mathbb{N}$, by the formula*

$$Q_m(x_1 \otimes x_2 \otimes \cdots \otimes x_m) = Q_m^1(x_1 \otimes \cdots \otimes x_m)\ + \sum_{i=1}^{m-1} \sum_{\sigma \in Sh(p,q)} \epsilon(\sigma)\ Q_i^1(x_{\sigma(1)} \otimes \cdots x_{\sigma(i)}) \otimes x_{\sigma(i+1)} \otimes \cdots \otimes x_{\sigma(m)}$$

Proof. The proposition follows directly from Lemma 2.4 in [LM95]. □

2.2 Differential-graded Lie algebras

Before we get into the details about L_∞-algebras, we are going to generalize the Lie algebra structure to graded vector spaces. This will not only help in illustrating how brackets work on graded vector spaces, but also provide a simple, non-trivial instance of $L_\infty - algebras$. We shall refer to [Ryv16] for the results of this section.

We know that a Lie algebra is a vector space equipped with a skew-symmetric, bilinear bracket $[\cdot, \cdot]$ which follows the Jacobi Identity:

$$[[x_1, x_2], x_3] - [[x_1, x_3], x_2] + [[x_2, x_3], x_1] = 0$$

We can write this as

$$\sum_{\sigma\in Sh(2,1)} (-1)^{\sigma}[[x_{\sigma(1)}, x_{\sigma(2)}], x_{\sigma(3)}] = 0$$

We'll now generalize this structure to graded vector spaces such that the Lie bracket respects the grading:

Definition 2.2.1. A graded vector space $V = \bigoplus_{i\in\mathbb{Z}} V_i$ equipped with a bilinear bracket $[\cdot,\cdot]$ is a **graded Lie algebra** if it follows the following conditions:

- $[V_i, V_j] \subset V_{i+j}$
- *Graded skew-symmetry*: For all homogeneous elements $x_1, x_2, x_3 \in V$, we have

$$[x_1, x_2] = -(-1)^{|x_1||x_2|}[x_2, x_1]$$

- *Graded Jacobi identity*:

$$[[x_1, x_2], x_3] - (-1)^{|x_2||x_3|}[[x_1, x_3], x_2] + (-1)^{|x_1||x_2|+|x_1||x_3|}[[x_2, x_3], x_1] = 0$$

The third condition can also be written like

$$\sum_{\sigma\in Sh(2,1)} (-1)^{\sigma}\epsilon(\sigma; x_1, x_2, x_3)[[x_{\sigma(1)}, x_{\sigma(2)}], x_{\sigma(3)}] = 0$$

by using the Koszul sign.

Example 2.2.2. We discuss an often-occurring example of graded Lie algebras: multi-vector fields (we refer to section 2.2 in [CFM21]). A multi-vector field of degree k on a manifold M is a section of $\bigwedge^k TM$. Here, the 0- vector fields are functions. Just like every vector field X can be identified with a derivation $\mathcal{L}_X : C^\infty(M) \to C^\infty(M)$, a k-vector field ϑ can be identified with a k-derivation

$$\begin{aligned} \mathcal{L}_\vartheta : C^\infty(M) \times \cdots \times C^\infty(M) &\to C^\infty(M) \\ (f_1, \ldots, f_k) &\mapsto \vartheta(df_1, \ldots, df_k) \end{aligned}$$

We denote the space of all k-vector fields as $\mathfrak{X}^k(M)$ and we define

$$\mathfrak{X}^\bullet(M) := \bigoplus_{k\geq 0} \mathfrak{X}^k(M)$$

Here we set the degree of the space of $(k+1)$-vector fields to be k. We turn this graded vector space into a graded Lie algebra by equipping it with the *Schouten bracket*, which is defined as follows: Given $\vartheta \in \mathfrak{X}^{k+1}(M)$, $\zeta \in \mathfrak{X}^{l+1}(M)$, their Schouten bracket $[\vartheta, \zeta] \in \mathfrak{X}^{k+l+1}(M)$ is the unique multivector field satisfying

$$\mathcal{L}_{[\vartheta,\zeta]} = \mathcal{L}_\vartheta \circ \mathcal{L}_\zeta - (-1)^{kl}\mathcal{L}_\zeta \circ \mathcal{L}_\vartheta$$

where

$$\mathcal{L}_\vartheta \circ \mathcal{L}_\zeta(f_1, \ldots, f_{k+l+1}) := \sum_{\sigma\in Sh(k,l+1)} (-1)^{\sigma}\mathcal{L}_\vartheta(f_{\sigma(1)}, \ldots, f_{\sigma(k)}, \mathcal{L}_\zeta(f_{\sigma(k+1)}, \ldots f_{\sigma(k+l+1)}))$$

For, example, consider $X_1, X_2, X_3 \in \mathfrak{X}(M)$, then

$$[X_1, X_2 \wedge X_3] = [X_1, X_2] \wedge X_3 + X_2 \wedge [X_1, X_3]$$

Generally, if we have decomposable multivector fields $\vartheta = X_0 \wedge \cdots \wedge X_k$ and $\zeta = Y_0 \wedge \cdots \wedge Y_l$, then

$$[\vartheta, \zeta] = \sum_{i=0}^{k} \sum_{j=0}^{l} (-1)^{i+j} [X_i, Y_j] \wedge X_0 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_k \wedge Y_0 \wedge \cdots \wedge \hat{Y}_j \wedge \cdots \wedge Y_l$$

This bilinear, graded skew-symmetric bracket turns $\mathfrak{X}^\bullet(M)$ into a graded Lie algebra.

Given a graded Lie algebra, we equip it with a differential $d : V \to V$, which satisfies the following conditions:

- $d(V_i) \subset V_{i+1}$
- $d^2 = 0$
- $d[x_1, x_2] = [dx_1, x_2] + (-1)^{|x_1||x_2|}[x_1, d(x_2)]$ (graded Leibniz rule)

Equipped with such a differential, V is called a *differential-graded Lie algebra*. We will now present the same definition, but rewrite the remaining equations using Koszul signs and summations over unshuffles.

Definition 2.2.3. A **differential-graded Lie algebra** is a graded vector space $V = \bigoplus_{i \in \mathbb{Z}} V_i$ equipped with skew-symmetric, linear functions $d : V \to V$ and $[\cdot, \cdot] : V^{\otimes 2} \to V$ of degrees 1 and 0 respectively, such that for all homogeneous elements $x_1, x_2, x_3 \in V$:

- $\sum\limits_{\sigma \in Sh(1,0)} (-1)^\sigma \epsilon(\sigma; x_1) d(d(x_{\sigma(1)})) = 0$
- $\sum\limits_{\sigma \in Sh(1,1)} (-1)^\sigma \epsilon(\sigma; x_1, x_2) [d(x_{\sigma(1)}), x_{\sigma(2)}] = \sum\limits_{\sigma \in Sh(2,0)} (-1)^\sigma \epsilon(\sigma; x_1, x_2) d[x_{\sigma(1)}, x_{\sigma(2)}]$
- $\sum\limits_{\sigma \in Sh(2,1)} (-1)^\sigma \epsilon(\sigma; x_1, x_2, x_3) [[x_{\sigma(1)}, x_{\sigma(2)}], x_{\sigma(3)}] = 0$

The identities look quite involved in this form, but note that the first identity is just $d^2 = 0$, the second one is the graded Leibniz rule, and the third is the graded Jacobi identity. The reason for expressing the identities in such a seemingly unnatural way will become clear shortly. In the meantime, it helps to keep in mind that all we have done is extend Lie algebras to graded vector spaces and equip them with a differential.
We now finally move on to describing L_∞-algebras.

2.3 L_∞-algebras as differential graded coalgebras

L_∞-algebras, in a general sense, are graded vector spaces equipped with a series of graded skew-symmetric multi-linear maps (brackets) which obey a generalized version of the Jacobi identity (or *identities*). Stating this definition right away might be counter-productive, so we start with a much simpler (and equivalent) definition.

For a given graded vector space V, we consider the coalgebra $C = \bar{S}(V)$. We start by describing a shifted version of this L_∞ structure, such that the degree of the brackets becomes 1. This is just given by equipping the coalgebra $\bar{S}(V)$ with a degree one coderivation. We shall follow [Sha14] as our main reference for the remainder of this chapter.

Definition 2.3.1. An $L_\infty[1]$ **structure** on a graded vector space V is a choice of a degree 1 coderivation Q on $\bar{S}(V)$

We know from Prop 1.1.10 that the coderivation Q is determined uniquely by a series of maps $m_k := Q_k^1 : \bar{S}^k(V) \to V$. The generalized Jacobi identity for these maps is then equivalent to $Q \circ Q = 0$. We state this explicitly in the following proposition:

Proposition 2.3.2. *An $L_\infty[1]$ structure in a graded vector space V uniquely determines a series of linear maps $m_k : \bar{S}^k(V) \to V,\ k \in \mathbb{N}$ such that*

$$\sum_{r+s=k} \sum_{\sigma \in Sh(r,s)} \epsilon(\sigma) m_{s+1}(m_r(x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(r)}) \otimes x_{\sigma(r+1)} \otimes \cdots \otimes x_{\sigma(k)}) = 0$$

where $\epsilon(\sigma) = \epsilon(\sigma; x_1, x_2, \cdots, x_k)$ and $x_1, x_2, \cdots x_k \in V$. Conversely, any such family of degree one linear maps uniquely determines a degree one coderivation on $\bar{S}(V)$.

Now, to obtain the definition for an L_∞ structure, we just shift the degree of the vector space, i.e.

Definition 2.3.3. An $\mathbf{L}_\infty$ **structure** on a graded vector space V is an $L_\infty[1]$ structure on V[1].

We call a graded vector space V with an L_∞ structure an L_∞ algebra. Now that we have the definition of an L_∞ algebra, we try to determine its brackets and the generalized Jacobi identity. We know by the décalage isomorphism that

$$\Lambda^k(V) = \Lambda^k(s(s^{-1}V)) \cong s^k \bar{S}^k(s^{-1}V)$$

Thus, we have $s^{-k} : \Lambda^k(V) \to \bar{S}^k(s^{-1}V)$. We now define $l_k : \Lambda^k(V) \to V$ such that it makes the following diagram commutative:

$$\begin{array}{ccc} \Lambda^k(V) & \xrightarrow{\quad l_k \quad} & V \\ {\scriptstyle s^{-k}}\downarrow & & \downarrow{\scriptstyle s^{-1}} \\ \bar{S}^k(s^{-1}V) & \xrightarrow[\quad m_k \quad]{} & s^{-1}V \end{array}$$

Note that since $|m_k| = 1\ \forall k \in \mathbb{N}$, we get that $|l_k| = 2 - k$. Using the identities for m_k in $V[1]$, we can obtain the analogous identities for l_k. We don't go through the entire proof, but an interested reader is welcome to check out [LS93].

Proposition 2.3.4. *An L_∞-algebra structure on a graded vector space uniquely determines a series of graded skew-symmetric multi-linear maps $\{l_k : V^{\otimes k} \to V | 1 \leq k < \infty\}$ with $|l_k| = 2 - k$ such that $\forall n \in \mathbb{N}$:*

$$\sum_{i+j=n+1} \sum_{\sigma \in Sh(i,n-1)} (-1)^{i(j-1)} \epsilon(\sigma)(-1)^{\sigma} l_j(l_i(x_{\sigma(1)}, \cdots, x_{\sigma(i)}), x_{\sigma(i+1)}, \cdots, x_{\sigma(n)}) = 0$$

Here, note that for $n = 1$ we get $l_1 \circ l_1 = 0$. Since $|l_1| = 1$, every L_∞ algebra V has an underlying cochain complex $(V, d = l_1)$

$$\cdots \xrightarrow{d} V_{i-1} \xrightarrow{d} V_i \xrightarrow{d} V_{i+1} \xrightarrow{d} \cdots$$

Denoting $l_2 = [\cdot, \cdot]$, we get

$$d[x_1, x_2] = [dx_1, x_2] + (-1)^{|x_1||x_2|}[x_1, d(x_2)]$$

i.e., l_2 can be interpreted as a graded skew-symmetric bracket. This bracket does not usually satisfy the Jacobi Identity. If we have $l_i = 0$ for $i \geq 3$, l_2 satisfies the graded Jacobi identity. In fact, the identities of the L_∞ algebra for $n \in \{1, 2, 3\}$ exactly match up with the three conditions in Definition 2.2.3. Thus, a differential graded Lie algebra is an L_∞-algebra for which $l_k = 0$ for $k \geq 3$.

Definition 2.3.5. An L_∞ algebra is said to be a **Lie n-algebra** if the underlying graded vector space is concentrated in the degrees $0, -1, \ldots, 1 - n$

Note that for a Lie n-algebra, $l_k = 0$ for $k > n + 1$ as $|l_k| = 2 - k$. Also, a Lie 1-algebra is just a Lie algebra.

Example 2.3.6 (Lie 2-algebras)**.** We consider another simple yet non-trivial example of L_∞ algebras. Lie 2-algebras consist of a complex

$$V_{-1} \xrightarrow{d} V_0$$

We denote $d := l_1$, $[\cdot, \cdot] := l_2$ and $J := l_3$. Note that $l_k = 0$ for $k \geq 4$.
Given $x_1, x_2, x_3, x_4 \in V_\bullet$, the generalized Jacobi identities that these maps satisfy are

$$\begin{aligned} d^2 &= 0 \\ d[x_1, x_2] &= [dx_1, x_2] + (-1)^{|x_1||x_2|}[x_1, d(x_2)] \\ [x_1, J(x_2, x_3, x_4)] + J(x_1, [x_2, x_3], x_4) + J(x_1, x_3, [x_2, x_4]) & \\ +[J(x_1, x_2, x_3), x_4] + [x_3, J(x_1, x_2, x_4)] &= J(x_1, x_2, [x_3, x_4]) + J([x_1, x_2], x_3, x_4) \\ &\quad + [x_2, J(x_1, x_3, x_4)] + J(x_2, [x_1, x_3], x_4) \\ &\quad + J(x_2, x_3, [x_1, x_4]) \end{aligned}$$

Cohomology of L_∞-Algebras

We consider the cohomology of the cochain complex underlying a L_∞-algebra V and show that the bracket l_2 satisfies the graded Jacobi identity over it.

Lemma 2.3.7. *Given an L_∞-algebra $(V, \{l_k\})$, the space of the cohomology classes of its underlying cochain complex is a graded Lie algebra.*

Proof. To prove this, we first show that l_2 is well-defined over the cohomology.

Consider $\overline{v_1}, \overline{v_2} \in H^i(V)$, where $\overline{v_k}$ denotes the cohomology class of v_k (note that this would imply that v_1, v_2 are closed, i.e., $dv_1 = dv_2 = 0$). Using the graded-Leibniz identity stated above, we get that for any $\omega \in V_{i-1}$

$$\begin{aligned}[v_1 + d\omega, v_2] &= [v_1, v_2] + [d\omega, v_2] \\ &= [v_1, v_2] + d[\omega, v_2] + (-1)^{|\omega||v_2|}[dv_2, \omega] \\ &= [v_1, v_2] + d[\omega, v_2]\end{aligned}$$

In the final inequality, we use the fact that $dv_2 = 0$. Thus, we see that

$$\overline{[v_1 + d\omega, v_2]} = \overline{[v_1, v_2]}$$

which implies that l_2 is well-defined over the cohomology. Now, to prove that it satisfies the Jacobi identity, we first note that the generalized Jacobi identity corresponding to $n = 3$ is

$$\begin{aligned}&(-1)^{|x_1||x_3|}\left[[x_1, x_2], x_3\right] + (-1)^{|x_2||x_3|}\left[[x_3, x_1], x_2\right] + (-1)^{|x_1||x_2|}\left[[x_2, x_3], x_1\right] \\ &= (-1)^{|x_1||x_3|+1}\left\{dl_3(x_1, x_2, x_3) + l_3(dx_1, x_2, x_3) + (-1)^{|x_1|}l_3(x_1, dx_2, x_3) + (-1)^{|x_1|+|x_2|}l_3(x_1, x_2, dx_3)\right\}\end{aligned}$$

If $\overline{x_1}, \overline{x_2}, \overline{x_3} \in H^i(V)$, then $dx_1 = dx_2 = dx_3 = 0$ and $\overline{dl_3(x_1, x_2, x_3)} = 0$ as it would be exact. Thus, over the cohomology $H^\bullet(V)$, the identity becomes

$$(-1)^{|x_1||x_3|}\left[[\overline{x_1}, \overline{x_2}], \overline{x_3}\right] + (-1)^{|x_2||x_3|}\left[[\overline{x_3}, \overline{x_1}], \overline{x_2}\right] + (-1)^{|x_1||x_2|}\left[[\overline{x_2}, \overline{x_3}], \overline{x_1}\right] = 0$$

□

Thus, the bracket l_2 turns the cohomology of V into a graded Lie algebra. Another way to see this is that a L_∞-algebra is a Lie algebra up to a chain homotopy generated by l_3. This is why L_∞-algebras are also referred to as *Strongly Homotopy Lie algebras*.

2.4 L_∞-morphisms

In this section, we will discuss morphisms between L_∞ algebras. These morphisms need to be defined in a way such that they preserve the algebraic structure of the L_∞ algebras. For this, it's best to start at the coalgebra structure underlying an L_∞ algebra. Given two L_∞ algebras $(L, \{l_k\})$ and $(L', \{l'_k\})$, consider their associated coalgebras $C = \bar{S}(L[1])$ and $C' = \bar{S}(L'[1])$ equipped with coderivations Q and Q', respectively as defined in Section 2.2.

Definition 2.4.1. An L_∞**-morphism** between L_∞-algebras $(L, \{l_k|\})$ and $(L', \{l'_k\})$ is a morphism between their differential graded coalgebras $F : (C, Q) \to (C', Q')$ such that

$$F \circ Q = Q' \circ F$$

To try to get the explicit expressions, we first define the following restrictions:

$$F_k^n = pr_{\bar{S}^n(L'[1])} \circ F|_{\bar{S}^k(L[1])} : \bar{S}^k(L[1]) \to \bar{S}^n(L'[1])$$

Now, similar to Prop 1.1.10 we claim that the morphism F is uniquely defined by

$$F^1 = F_1^1 + F_2^1 + F_3^1 + \cdots$$

The proof of this claim can be found in Appendix A2 of [CFRZ16]

Now, we define a collection of linear maps $\{f_k\}$ with $f_k : L^{\otimes k} \to L'$ such that it makes the following diagram commutative:

$$\begin{array}{ccc} \Lambda^k(L) & \xrightarrow{\quad f_k \quad} & L' \\ \downarrow{\scriptstyle s^{-k}} & & \downarrow{\scriptstyle s^{-1}} \\ \bar{S}^k(s^{-1}L) & \xrightarrow[F_k^1]{\qquad} & s^{-1}L' \end{array}$$

Now, the condition of compatibility $\{f_k\}$ follows is now equivalent to

$$\sum_{k=1}^{m} F_k^1 Q_m^k = \sum_{k=1}^{m} Q_k'^1 F_m^k \;\; \forall m \in \mathbb{N}$$

Example 2.4.2 (5.3 in [Zam12])**.** Let $(V, d, [\cdot,\cdot], J)$ and $(V', d', [\cdot,\cdot]', J')$ be Lie 2-algebras. An L_∞ morphism $\phi : V \to V'$ for this case would be a collection of linear maps:

$$\begin{aligned} \phi_0 &: V_0 \to V_0' \\ \phi_1 &: V_{-1} \to V_{-1}' \\ \phi_2 &: \bigwedge^2 V_0 \to V_{-1}' \end{aligned}$$

such that

$$\begin{aligned} d' \circ \phi_1 &= \phi_0 \circ d \\ d'(\phi_2(x,y)) &= \phi_0([x,y]) - [\phi_0(x), \phi_0(y)]' \;\text{ for all } x, y \in V_0 \\ \phi_2(df, y) &= \phi_1([f,y]) - [\phi_1(f), \phi_0(y)]' \;\text{ for all } f \in V_{-1},\; y \in V_0 \end{aligned}$$

and for $x, y, z \in V_0$,

$$\begin{aligned} \phi_0(J(x,y,z)) - J'(\phi_0(x), \phi_0(y), \phi_0(z)) &= \phi_2(x,[y,z]) - \phi_2(y,[x,z]) + \phi_0(z,[x,y]) \\ &\quad + [\phi_0(x), \phi_2(y,z)]' - [\phi_0(y), \phi(x,z)]' + [\phi_0(z), \phi_2(x,y)]' \end{aligned}$$

Definition 2.4.3. A **strict L_∞-morphism** between L_∞-algebras $(L, \{l_k|\})$ and $(L', \{l'_k\})$ is a degree 0 linear map $f : L \to L'$ satisfying

$$l'_k \circ f^{\otimes k} = f \circ l_k$$

For these *strict* L_∞-morphisms, the corresponding morphism $F : (C, Q) \to (C', Q')$ satisfies

$$F_k^1 = 0 \;\; \forall k \geq 2$$

Finally, we define the concept of a *quasi-isomorphism*, which is a morphism that induces an isomorphism on the cohomology of the cochain structure underlying the L_∞ algebras:

Definition 2.4.4. An L_∞-morphism $(f_k)_{k \in \mathbb{N}} : (L, \{l_k\}) \to (L', \{l'_k\})$ is called a **quasi-isomorphism** if and only if $f_1 : (L, l_1) \to (L', l'_1)$ induces an isomorphism on the cohomology of the underlying complexes:

$$H(f_1) : H^\bullet(L) \xrightarrow{\cong} H^\bullet(L')$$

Morphisms from Lie algebras to Lie n-algebras

Since the expressions for a general L_∞ morphism are quite complicated, we instead look at a simpler case in this section: a morphism from a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ to a Lie n-algebra (L, l_k)which follows the following property:

$$\forall k \geq 2, \quad l_k(x_1, \dots, x_k) = 0 \qquad \text{if} \sum_{i=1}^{k} |x_i| < 0 \tag{2.1}$$

This shall also be useful to us in the next chapter, as this property is followed by the Lie n-algebras arising from multisymplectic manifolds.

Proposition 2.4.5 (Proposition 3.8 in [CFRZ16])**.** *If $(\mathfrak{g}, [\cdot,\cdot])$ is Lie algebra and (L, l_k) a Lie n-algebra following property* (2.1)*, then the graded skew-symmetric maps*

$$f_k : \mathfrak{g}^{\otimes k} \to L, \quad |f_k| = 1 - k, \quad 1 \leq k \leq n$$

are the components of an L_∞-morphism $(f_k)_k : \mathfrak{g} \to L$ if and only if $\forall x_i \in \mathfrak{g}$

$$\sum_{1 \leq i < j \leq k} (-1)^{i+j+1} f_{k-1}([x_i, x_j], x_1, \dots, \hat{x_i}, \dots, \hat{x_j}, \dots, x_k) = l_1 f_k(x_1, \dots, x_k) + l_k(f_1(x_1), \dots, f_1(x_k))$$

$$\sum_{1 \leq i < j \leq n+1} (-1)^{i+j+1} f_n([x_i, x_j], x_1, \dots, \hat{x_i}, \dots, \hat{x_j}, \dots, x_{n+1}) = l_{n+1}(f_1(x_1), \dots, f_1(x_{n+1}))$$

Proof. Refer to Corollary A8 in [CFRZ16] □

Chapter 3

Multisymplectic geometry

Now that we are familiar with L_∞-algebras, we can begin to describe multisymplectic manifolds and their associated observables. Just as symplectic manifolds provide the geometric setting for Hamiltonian dynamics in classical mechanics, modeling phase spaces with a symplectic 2-form, multisymplectic manifolds serve as the geometric setting for covariant Hamiltonian formulations of classical field theories, where fields replace particles and the phase space becomes a multiphase space. They generalize symplectic manifolds by considering closed, non-degenerate differential forms of higher degree. We won't be delving too deep into field theory in this thesis, but some background on the role of multisymplectic geometry in classical field theory can be found in Section 2 of [RW19].

3.1 Multisymplectic manifolds

In this section, we'll go through various definitions generalizing symplectic manifolds, Hamiltonian vector fields, and so on. We shall mostly refer to [CFRZ16] for this section.

Definition 3.1.1. An (n+1) form ω on a smooth manifold M is said to be **n-plectic** if it is

1. closed: $d\omega = 0$
2. non-degenerate: $\iota_v \omega = 0 \implies v = 0 \quad \forall x \in M, v \in T_x M$

The pair (M, ω) is said to be an n-plectic manifold. Note that a 1-plectic manifold is just a symplectic manifold.

The non-degeneracy of the form ω implies that the map $\iota_\bullet \omega : TM \to \bigwedge^n T^*M$ is injective. In the symplectic case, as the dimension of $\bigwedge^1 T^*M$ is the same as that of TM, we can argue that the map is an isomorphism. However, in the general case, this map is only an injection.

Example 3.1.2. Let M be a manifold equipped with a volume form $\omega \in \Omega^{n+1}(M)$. Then (M, ω) is an n-plectic manifold.

Remark 3.1.3. Another such case where the map $\iota_\bullet \omega : TM \to \bigwedge^n TM$ is an isomorphism is when $\omega \in \Omega^{n+1}(M)$ is a volume form. Note that since the manifold is of dimension $(n+1)$, the vector bundle $\bigwedge^n TM$ has fibers of dimension 1. We can then argue, similar to the symplectic case, that the map becomes an isomorphism.

Example 3.1.4. Let Q be a manifold (of dimension $N \geq n$) and consider $M = \bigwedge^n T^*Q$, i.e., a multi-cotangent bundle of Q. Such a manifold is always equipped with a tautological form $\theta \in \Omega^n(M)$ defined by

$$\theta_{\eta_y}(v_1, \dots, v_n) = \eta_y(\pi_* v_1, \dots, \pi_* v_n)$$

$\forall y \in Q, \eta_y \in \bigwedge^n(T^*Q), v_i \in T_{\eta_y}(\bigwedge^n T^*Q)$ and $\pi : \Lambda^n(T^*M) \to M$.
Then, $(M, -d\theta)$ is an n-plectic manifold. Similar to the symplectic case, we express θ in local coordinates [RW19]. For $U \subset Q$, consider $\bar{y_1}, \dots, \bar{y_N}$ a system of coordinates. Then it induces a system of coordinates (y_i, p^I) on $\bigwedge^n T^*Q|_U$, for $1 \leq i \leq N$ and $I \in \binom{N}{n}$. Here $I = (i_1, \dots, i_n)$ is a set of strictly ascending multi-indices, $y_i = \bar{y_i} \circ \pi$ and for

$$\begin{aligned} p^I : \bigwedge^n T^*Q|_U &\to \mathbb{R} \\ \zeta &\mapsto \zeta\left(\frac{\partial}{\partial y_{i_1}}, \dots, \frac{\partial}{\partial y_{i_n}}\right) \end{aligned}$$

Similar to Example 1.1.7, we can now carry out a similar computation for some $\zeta = \sum_I a^I (d\bar{y_I})_x \in \bigwedge^n T^*_x Q$ and $v \in T_\zeta(\bigwedge^n T^*Q)$ where $d\bar{y_I} = d\bar{y_{i_1}} \wedge \cdots \wedge d\bar{y_{i_n}}$ to obtain that $\theta = \sum_I p^I dy_I$. So

$$\omega = -\sum_I dp^I \wedge dy_I$$

Example 3.1.5 (Compact simple Lie groups)**.** Every simple Lie group has a 2-plectic structure. Given a simple Lie group G, consider an invariant inner product $\langle \cdot, \cdot \rangle$ on its associated Lie algebra $\mathfrak{g}$. Note that for a finite-dimensional Lie algebra $\mathfrak{g}$, we have the invariant, symmetric, bilinear form given by

$$K(x, y) = \text{trace}(ad(x) \circ ad(y))$$

where $ad(x)(y) = [x, y]$. This inner product is known as the **Killing form** on $\mathfrak{g}$. Thus, for the cases of concern to us, the Lie algebra always has such an invariant inner product. Given $x, y, z \in \mathfrak{g}$, we define the following 3-form on $\mathfrak{g}$:

$$\theta(x, y, z) = \langle x, [y, z] \rangle$$

Denote by $L_g : G \to G$ the left translation map, then we can define for $v_1, v_2, v_3 \in T_g G$:

$$\nu_g(v_1, v_2, v_3) := \theta(L_{g^{-1}*} v_1,\ L_{g^{-1}*} v_2,\ L_{g^{-1}*} v_3)$$

We now show that ν is a skew-symmetric 2-form on G.

Claim: $\nu \in \Omega^3(G)$

Proof. Note that $\theta(x, [y, z]) = -\theta(x, [z, y])$ for all $x, y, z \in \mathfrak{g}$ as $[\cdot, \cdot]$ is skew-symmetric. Moreover, as the inner product $\langle \cdot, \cdot \rangle$ is invariant, we can say that for all $x, y, z \in \mathfrak{g}$

$$\langle x, z \rangle = \langle Ad_{\rho(t)} x,\ Ad_{\rho(t)} z \rangle$$

where $Ad : G \to GL(\mathfrak{g})$ is the adjoint representation of G and $\rho(t) := exp(ty)$. Taking a derivative with respect to t, we get

$$0 = \langle ad_y x,\ z\rangle + \langle x,\ ad_y z\rangle$$

where $ad : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is the adjoint representation of $\mathfrak{g}$. Note that $ad_y x = [y, x]$, thus we have that

$$\langle z, [y, x]\rangle = -\langle x, [y, z]\rangle$$

Thus, θ is skew-symmetric, which implies that $\nu \in \Omega^3(G)$.

□

Now, we see by definition that ν is left-invariant. Moreover, as $\langle \cdot, \cdot\rangle$ is invariant with respect to the adjoint action: $Ad_g = L_{g*} \circ R_{g^{-1}*}$, we get that ν is also a right-invariant form. We show that this implies that ν is closed.

Claim: $d\nu = 0$

Proof. Consider $A := \nu_e$ as an invariant form on $\mathfrak{g}$. It suffices to prove that $d_{\mathfrak{g}} A = 0$. For $v_1, v_2, v_3, v_4 \in \mathfrak{g}$, we have

$$\begin{aligned} d_{\mathfrak{g}} A(v_1, v_2, v_3, v_4) &= \sum_{1\le i<j\le 4} (-1)^{i+j} A([v_i, v_j], v_1, \ldots, \hat{v}_i, \ldots \hat{v}_j, \ldots v_4) \\ &= -A([v_1, v_2], v_3, v_4) + A([v_1, v_3], v_2, v_4) - A([v_1, v_4], v_2, v_3) \\ &\quad - A([v_2, v_3], v_1, v_4) + A([v_2, v_4], v_1, v_3) - A([v_3, v_4], v_1, v_2) \end{aligned}$$

Note that since A is invariant with respect to the adjoint action, we get

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} A(v_2, v_3, v_4) &= \frac{d}{dt}\Big|_{t=0} A(Ad_{exp(tv_1)} v_2, Ad_{exp(tv_1)} v_3, Ad_{exp(tv_1)} v_4) \\ &= A(ad_{v_1} v_2, v_3, v_4) + A(v_2, ad_{v_1} v_3, v_4) + A(v_2, v_3, ad_{v_1} v_4) \\ &= A([v_1, v_2], v_3, v_4) + A(v_2, [v_1, v_3], v_4) + A(v_2, v_3, [v_1, v_4]) \\ &= A([v_1, v_2], v_3, v_4) - A([v_1, v_3], v_2, v_4) + A([v_1, v_4], v_2, v_3) \end{aligned}$$

Since the left-hand side doesn't depend on the variable t, we get that

$$A([v_1, v_2], v_3, v_4) - A([v_1, v_3], v_2, v_4) + A([v_1, v_4], v_2, v_3) = 0 \tag{3.1}$$

Switching the indices in the above expression, we also get

$$-A([v_1, v_2], v_3, v_4) - A([v_2, v_3], v_1, v_4) + A([v_2, v_4], v_1, v_3) = 0 \tag{3.2}$$
$$-A([v_1, v_3], v_2, v_4) + A([v_2, v_3], v_1, v_4) + A([v_3, v_4], v_1, v_2) = 0 \tag{3.3}$$
$$-A([v_1, v_4], v_2, v_3) + A([v_2, v_4], v_1, v_3) - A([v_3, v_4], v_1, v_2) = 0 \tag{3.4}$$

We add/ subtract the left-hand and right-hand sides of the above equation as follows: $-(3.1) + (3.2) - (3.3) + (3.4)$. We get

$$2 \cdot d_{\mathfrak{g}} A(v_1, v_2, v_3, v_4) = 0$$

Thus, we get that $d_{\mathfrak{g}} A = 0$, which proves the claim. Thus, $d\nu = 0$ □

Moreover, we can show that it is non-degenerate. Consider $x \in \mathfrak{g}$ such that $\langle x, [y,z]\rangle = 0$ for all $y, z \in \mathfrak{g}$. Since G is simple, we have that $H^1(\mathfrak{g}) = 0$, i.e. $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. Thus, there exist $y_1, \ldots, y_n, z_1, \ldots, z_n \in \mathfrak{g}$ for some $n \in \mathbb{N}$ such that $x = \sum_i [y_i, z_i]$. By our assumption, we have that

$$\begin{aligned} \sum_i \langle x, [y_i, z_i]\rangle &= 0 \\ \implies \left\langle x, \sum_i [y_i, z_i] \right\rangle &= 0 \\ \implies \langle x, x\rangle &= 0 \\ \implies x &= 0 \end{aligned}$$

Since $x \in \mathfrak{g}$ was arbitrary, we have that θ is a non-degenerate form on $\mathfrak{g}$. Thus, we have that ν is a non-degenerate, closed 3-form i.e., it is a 2-plectic form on G. For further details, refer to section 4 of [BR10].

Definition 3.1.6. Let (M, ω) be an n-plectic manifold. An $(n-1)$-form $\beta \in \Omega^{n-1}(M)$ is said to be **Hamiltonian** if there exists $X_\beta \in \mathfrak{X}(M)$ such that

$$\iota_{X_\beta}\omega = -d\beta$$

The vector field X_β is said to be the Hamiltonian vector field corresponding to β.

The space of $(n-1)$ Hamiltonian forms and Hamiltonian vector fields is denoted by $\Omega^{n-1}_{Ham}(M)$ and $\mathfrak{X}_{Ham}(M)$ respectively.
Note that in the symplectic case, we have that $C^\infty_{Ham}(M) = C^\infty(M)$ since the symplectic form induces an isomorphism $\iota_\bullet \omega : TM \to \bigwedge^1 T^*M : v \mapsto \iota_v \omega$. However, this is not true generally.

Definition 3.1.7. Let (M, ω) be an n-plectic manifold. A vector field $X \in \mathfrak{X}(M)$ is said to be a **local Hamiltonian vector field** iff $\mathcal{L}_X \omega = 0$.

We denote the space of local Hamiltonian vector fields of M as $\mathfrak{X}_{LHam}(M)$.
Note that if X_α is a Hamiltonian vector field corresponding to $\alpha \in \Omega^{n-1}_{Ham}(M)$, then

$$\begin{aligned} \mathcal{L}_{X_\alpha}\omega &= \iota_{X_\alpha} d\omega + d\iota_{X_\alpha}\omega \\ &= 0 + d(-d\alpha) = 0 \end{aligned}$$

Thus, $\mathfrak{X}_{Ham}(M) \subseteq \mathfrak{X}_{LHam}(M)$.

Proposition 3.1.8 (Lemma 4.5 in [CFRZ16])**.** *Let (M, ω) be an n-plectic manifold and $v_1, v_2 \in \mathfrak{X}_{LHam}(M)$, then we have that $[v_1, v_2] \in \mathfrak{X}_{Ham}(M)$ with $\iota_{[v_1,v_2]}\omega = -d(\iota_{v_2 \wedge v_1}\omega)$*

Proof. Let $v_1, v_2 \in \mathfrak{X}_{LHam}(M)$. Then we have by Cartan that

$$\begin{aligned} \iota_{[v_1,v_2]}\omega &= \mathcal{L}_{v_1}\iota_{v_2}\omega - \iota_{v_1}\mathcal{L}_{v_2}\omega \\ &= \mathcal{L}_{v_1}\iota_{v_2}\omega \\ &= d\iota_{v_1}\iota_{v_2}\omega + \iota_{v_1} d\iota_{v_2}\omega \\ &= d\iota_{v_1}\iota_{v_2}\omega + \iota_{v_1}(\mathcal{L}_{v_2}\omega - \iota_{v_2} d\omega) \\ &= d\iota_{v_1}\iota_{v_2}\omega \end{aligned}$$

Thus, we have that $\iota_{[v_1,v_2]}\omega = d\iota_{v_2 \wedge v_1}\omega$. □

3.2 Lie n-algebra of observables

Similar to the case of symplectic manifolds, the space of $(n-1)$ Hamiltonian forms of an n-plectic manifold can be endowed with a skew-symmetric binary bracket defined as

$$\{\alpha, \beta\} = \iota_{X_\beta}\iota_{X_\alpha}\omega \qquad \alpha, \beta \in \Omega^{n-1}_{Ham}(M)$$

This bracket does not generally satisfy the Jacobi identity and thus isn't a Lie bracket. Refer to Example 3.2.1 below for the proof of this statement in the setting of 2-plectic manifolds. This lack of a sincere Lie bracket on the space of Hamiltonian forms is why L_∞ algebras came into place. In this section, we'll be constructing the generalization of the 'Lie algebra of Observables' on multisymplectic manifolds.

Let (M, ω) be an n-plectic manifold. Consider the following graded vector space:

$$\begin{aligned} L^0 &= \Omega^{n-1}_{Ham}(M) \\ L^i &= \Omega^{n-1+i}(M) \quad 1-n \leq i < 0 \end{aligned}$$

along with the multi-brackets $l_k : L^{\otimes^k} \to L$ defined as

$$l_1(\alpha) = \begin{cases} 0 & \text{if } |\alpha| = 0 \\ d\alpha & otherwise \end{cases}$$

and for $k \geq 2$,

$$\begin{aligned} l_k(\alpha_1, \ldots, \alpha_k) &= \varsigma(k)\ \iota_{X_{\alpha_k}}\iota_{X_{\alpha_{k-1}}}\ldots\iota_{X_1}\omega \quad \text{if } |\alpha_1 \otimes \cdots \otimes \alpha_k| = 0 \\ &= 0 \qquad\qquad \text{otherwise} \end{aligned}$$

where $\varsigma(k) := (-1)^{\frac{k(k+1)}{2}}$. The multi-brackets defined above satisfy the generalized Jacobi identities defined in section 2.2. A proof of this statement can be found in section 5 of [Rog12].
Thus, the graded vector space is a Lie n-algebra. It is denoted as $Ham_\infty(M, \omega)$ and known as the 'Lie n-algebra of observables'.

Example 3.2.1 (2-plectic manifolds). We now examine the simplest nontrivial instance of a multisymplectic manifold: the 2-plectic case, and the corresponding Lie 2-algebra of observables. Let (M, ω) be a 2-plectic manifold. The space of observables is a 2-term complex:

$$C^\infty(M) \xrightarrow{d} \Omega^1_{Ham}(M)$$

where $\Omega^1_{Ham}(M)$ denotes the space of Hamiltonian 1-forms.
We denote the structure maps of the associated Lie 2-algebra as follows:

$$\begin{aligned} d &:= l_1 : L \to L \\ [\cdot, \cdot] &:= l_2 : L^{\otimes 2} \to L : (\alpha_1, \alpha_2) \mapsto -\iota_{X_{\alpha_2}}\iota_{X_{\alpha_1}}\omega \\ J &:= l_3 : L^{\otimes 3} \to L : (\alpha_1, \alpha_2, \alpha_3) \mapsto \iota_{X_{\alpha_3}}\iota_{X_{\alpha_2}}\iota_{X_{\alpha_1}}\omega \end{aligned}$$

We can show that the Jacobi identity fails to hold in this case.
Let $\alpha_1, \alpha_2, \alpha_3 \in \Omega^1_{Ham}(M)$ and $X_1, X_2, X_3 \in \mathfrak{X}_{Ham}(M)$ be their respective Hamiltonian

vector fields. As $\mathcal{L}_{X_i}\omega = 0$ for $i = 1, 2, 3$ and ω is closed, the argument is pretty similar to the one we gave in the proof of Lemma 1.1.11, except that $d\iota_{X_3}\iota_{X_2}\iota_{X_1}\omega = dJ(\alpha_1, \alpha_2, \alpha_3) \neq 0$ as ω is a 3-form in this case. The rest of the argument follows similarly and we get

$$\begin{aligned} &\iota_{X_3}\iota_{[X_1,X_2]}\omega - \iota_{X_2}\iota_{[X_1,X_3]}\omega - \iota_{[X_2,X_3]}\iota_{X_1}\omega - d\iota_{X_3}\iota_{X_2}\iota_{X_1}\omega = 0 \\ \implies &[[\alpha_1, \alpha_2], \alpha_3] - [[\alpha_1, \alpha_3], \alpha_2] - [\alpha_1, [\alpha_2, \alpha_3]] - dJ(\alpha_1, \alpha_2, \alpha_3) = 0 \end{aligned}$$

Thus, we get that

$$[[\alpha_1, \alpha_2], \alpha_3] - [[\alpha_1, \alpha_3], \alpha_2] - [\alpha_1, [\alpha_2, \alpha_3]] = dJ(\alpha_1, \alpha_2, \alpha_3)$$

which is one of the generalized Jacobi identities we get for 2-plectic manifolds. For the remaining identities for 2-plectic manifolds, refer to Example 2.3.6.

Note that the graded Leibniz rule we get for $n = 2$ holds trivially in the case of Lie n-algebras of observables.

3.3 A digression on field theory

As mentioned earlier, multisymplectic geometry is closely related to the Hamiltonian formulation of classical field theories. It seems fitting to have an example that is closely related to field theory.

We consider spacetime as a 4-dimensional manifold M. The multi-phase space associated with it is given by $\Lambda^4 T^*M$, and as we mentioned in example 3.1.3, it is also endowed with a tautological form $\theta \in \Omega^4(\Lambda^4 T^*M)$. We consider $(\Lambda^4 T^*M, d\theta)$ as a 4-plectic manifold. A solution to the Euler-Lagrange equations for any system is represented by the image of a section $\xi \in \Gamma(\Lambda^4 T^*M)$. Given a Cauchy slice $C \subset \xi(M)$ and $\alpha \in \Omega^3_{Ham}(\Lambda^4 T^*M)$, the 'measurement' of α over C is

$$\int_C i_C^*\alpha \in \mathbb{R}$$

We have the following proposition:

Proposition 3.3.1 (Theorem 3.41 in [MC10]). *Given the multisymplectic manifold $(\Lambda^4 T^*M, \omega)$ where $\omega := -d\theta$ and $i_\xi : \xi(M) \hookrightarrow \Lambda^4 T^*M$, we have that for all $X \in \mathfrak{X}(M)$*

$$i_\xi^*(\iota_X\omega) = 0$$

Using the above proposition, we can see that if we have two homologous Cauchy slices $C_1, C_2 \subset \xi$, i.e., there exists an open subset $\Omega \subset \xi(M)$ such that $\partial\Omega = C_1 - C_2$, then

$$\int_{C_1} i^*\alpha - \int_{C_2} i^*\alpha = \int_{\partial\Omega} i^*\alpha = \int_\Omega i^*d\alpha = \int_\Omega i^*\iota_X\omega = 0$$

where i is the inclusion as given in the above proposition. Thus, we see that

$$\int_{C_1} i^*\alpha = \int_{C_2} i^*\alpha$$

i.e., this 'measurement' is unique up to the homology class of the Cauchy slice. We thank Prof. Frédéric Hélein for showing us the above result.

Chapter 4

Homotopy Moment Maps

In this chapter, we investigate the notion of a homotopy moment map, which serves as a natural generalization of the classical moment map (as discussed in Section 1.2) in the multisymplectic setting. As in symplectic geometry, one is often interested in Lie group actions that preserve the underlying geometric structure—in this case, a closed, non-degenerate $(n+1)$-form ω on a manifold M. When the infinitesimal generators of such an action are Hamiltonian vector fields, the action is termed 'Hamiltonian'. In symplectic geometry, a Hamiltonian action is accompanied by a moment map, which encodes the action of the Lie algebra $\mathfrak{g}$ in terms of functions on the manifold and satisfies a compatibility condition with the Poisson bracket.

However, in the multisymplectic setting, the observables no longer form a Lie algebra, but rather an L_∞-algebra. Accordingly, the classical notion of a moment map must be replaced by a more flexible structure that captures this higher algebraic complexity. A homotopy moment map is precisely such an object: it is an L_∞-morphism from the Lie algebra $\mathfrak{g}$ (viewed as an L_∞-algebra concentrated in degree zero) into the L_∞-algebra of observables on the multisymplectic manifold.

The goal of this chapter is to define homotopy moment maps and study their properties in detail. We begin by recalling the classical concept of moment maps to motivate the higher analog. We then introduce the definition of a homotopy moment map, describe the conditions under which such maps can exist, and interpret them in terms of symmetry and conservation laws in the multisymplectic framework. The chapter concludes with a discussion of concrete examples where homotopy moment maps can be constructed explicitly, providing insight into their structure and role in higher symplectic geometry.

Definition 4.0.1 (Definition/Proposition 5.1 in [CFRZ16])**.** Consider a Lie group G acting on an n-plectic manifold (M, ω) which preserves the multisymplectic form ω. Suppose the infinitesimal action $v : \mathfrak{g} \to \mathfrak{X}(M)$ is via Hamiltonian vector fields, where $\mathfrak{g}$ is the corresponding Lie algebra of G. A **homotopy moment map** $\mu : \mathfrak{g} \to Ham_\infty(M)$ is a lift of the Lie algebra morphism v via the L_∞ morphism π

$$\begin{array}{ccc} & & Ham_\infty(M) \\ & \nearrow & \downarrow \pi \\ \mathfrak{g} & \xrightarrow{\nu} & \mathfrak{X}(M) \end{array}$$

such that

$$d\mu_1 = -\iota_{v_x}\omega$$

$\mu : \mathfrak{g} \to Ham_\infty(M)$ being an L_∞-morphism means that by Proposition 2.4.5, we have a series of linear maps $\{\mu_i\}_i$, $\mu_k : \bigwedge^k \mathfrak{g} \to \Omega^{n-k}(M)$ such that for $2 \leq k \leq n$

$$\sum_{1\leq i<j\leq k} (-1)^{i+j+1}\mu_{k-1}([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_k) = d\mu_k(x_1, \dots, x_k) + \zeta(k)\iota(v_{x_1} \wedge \cdots \wedge v_{x_k})\omega$$

$$\sum_{1\leq i<j\leq n+1} (-1)^{i+j+1}\mu_n([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{n+1}) = \zeta(n+1)\iota(v_{x_1} \wedge \cdots \wedge v_{x_{n+1}})\omega$$

Note that when (M, ω) is a 1-plectic (i.e., a symplectic) manifold, the commutative diagram above turns into the diagram for comoment maps we see in Section 1.1.

Example 4.0.2 (Hamiltonian actions on 2-plectic manifolds)**.** Let's examine how homotopy moment maps function in a lower-degree setting. We consider that the Lie group G acts on the 2-plectic manifold (M, ω) by Hamiltonian vector fields. A homotopy moment map for this action would be a collection of maps

$$f_1 : \mathfrak{g} \to \Omega^1_{Ham}(M) \qquad\qquad f_2 : \bigwedge^2 \mathfrak{g} \to C^\infty(M)$$

such that

1. v_x is the Hamiltonian vector field of $f_1(x)$ for all $x \in \mathfrak{g}$

2. the following equations hold:

$$f_1([x, y]) - \{f_1(x), f_1(y)\} = df_2(x, y) \tag{4.1}$$

$$-l_3(f_1(x), f_1(y), f_1(z)) = f_2(x, [y, z]) - f_2(y, [x, z]) + f_2(z, [x, y]) \tag{4.2}$$

We know by Proposition 1.1.14 that in the symplectic case, any symplectic action by a semi-simple, connected Lie group G will admit a moment map. To extend this result to the 2-plectic case, we have to add another condition:

Proposition 4.0.3 (Proposition 7.1 in [CFRZ16])**.** *If G is a compact connected, semi-simple Lie group acting multisymplectically on a 2-plectic manifold (M, ω) and for all $x \in \mathfrak{g}$, there is a point $p \in M$ such that $v_x(p) = 0$, then there exists an equivariant homotopy moment map.*

We can comment on the uniqueness of the homotopy moment maps in this case. Similar to the symplectic case, some constraints of the cohomology of the Lie algebra do ensure the uniqueness of the homotopy moment map, but only up to a certain equivalence relation.

Proposition 4.0.4 (Proposition 7.5 (b) in [CFRZ16])**.** *If G is a compact connected, semi-simple Lie group acting multisymplectically on a 2-plectic manifold (M, ω), or more generally, if $H^1(\mathfrak{g}) = H^2(\mathfrak{g}) = 0$, then any two homotopy moment maps (f_1, f_2) and $(\tilde{f}_1, \tilde{f}_2)$ will be related as follows:*

$$\tilde{f}_1 = f_1 + d\psi$$

$$\tilde{f}_2(x, y) = f_2(x, y) + \psi([x, y])$$

for some linear map $\psi : \mathfrak{g} \to C^\infty(M)$.

Proof. We need to show that, given two moment maps with components (f_1, f_2) and $(\tilde{f}_1, \tilde{f}_2)$, there exists a map $\psi\colon \mathfrak{g} \to C^\infty(M)$ such that

$$\tilde{f}_1 - f_1 = d\psi \quad \text{and} \quad (\tilde{f}_2 - f_2)(x, y) = \psi([x, y]).$$

For x, y in $\mathfrak{g}$, we can in-fact define $\psi([x, y]) = (\tilde{f}_2 - f_2)(x, y)$. Since $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$, we have that for all $z \in \mathfrak{g}$, there exist $x_1, \ldots, x_n, y_1, \ldots, y_n \in \mathfrak{g}$ for some $n \in \mathbb{N}$ such that $z = \sum_i [x_i, y_i]$. Thus $\psi : \mathfrak{g} \to C^\infty(M)$ is well-defined.

Using equation (4.2), we can see that $(f_2 - \tilde{f}_2)$ is a cocycle in $H^2(\mathfrak{g}, C^\infty(M)) = H^2(\mathfrak{g}) \otimes C^\infty(M) = 0$. Thus, there exists a function $\psi : \mathfrak{g} \to C^\infty(M)$ such that $(f_2 - \tilde{f}_2) = d_\mathfrak{g}\psi$ i.e.,

$$(\tilde{f}_2 - f_2)(x, y) = \psi([x, y])$$

for all $x, y \in \mathfrak{g}$.
To complete the argument, we must ensure that the identity $\tilde{f}_1 - f_1 = d\psi$ holds. This identity is satisfied when both sides are evaluated on elements of the form $[x, y] \in \mathfrak{g}$, by eq.(4.1) and the preceding discussion. Since $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$, the equality holds for all elements of $\mathfrak{g}$. □

In the following sections, we will present and prove several examples of group actions that admit homotopy moment maps. To do so, however, we must first develop some necessary tools.

4.1 Encoding moment maps via a double complex

A homotopy moment map is a series of linear maps from $\Lambda^{\geq 1}\mathfrak{g}$ to $\Omega^\bullet(M)$. Using the identification $Hom(V, W) \cong V^* \otimes W$, it makes sense to consider the double complex $\Lambda^{\geq 1}\mathfrak{g}^* \otimes \Omega^\bullet(M)$. Note that $\Lambda^\bullet\mathfrak{g}^*$ is the Chevalley-Eilenberg complex. We can equip it with the differential

$$d_\mathfrak{g} : \bigwedge^k \mathfrak{g}^* \to \bigwedge^{k+1} \mathfrak{g}^*$$
$$\phi \mapsto \phi \circ \partial$$

Where $\partial : \bigwedge^k \mathfrak{g} \to \bigwedge^{k-1} \mathfrak{g}$ is the boundary operator on $\Lambda^\bullet\mathfrak{g}$ given by

$$\partial(x_1 \wedge \ldots \wedge x_k) = \sum_{1 \leq i < j \leq k} (-1)^{i+j}[x_i, x_j] \wedge x_1 \ldots \wedge \hat{x_i} \wedge \ldots \wedge \hat{x_j} \wedge \ldots \wedge x_k$$

For this new complex, the set of all elements of degree $k \in \mathbb{N}$ is given by

$$\bigoplus_{i=1}^{k-1} \Lambda^i\mathfrak{g}^* \otimes \Omega^{k-i}(M)$$

We equip the bi-complex with the differential $d_{tot} = d_\mathfrak{g} \otimes 1 + 1 \otimes d$. Note that here, $d_\mathfrak{g}$ is the Chevalley-Eilenberg differential equipped to the complex $\Lambda^{\geq 1}\mathfrak{g}^*$ and d is the deRham differential. To take the grading into account, we use the Koszul sign convention. Thus, on an element of $\Lambda^k\mathfrak{g}^* \otimes \Omega^\bullet(M)$, d_{tot} acts as $d_g + (-1)^k d$.

In this section, we focus on translating the conditions for the existence of homotopy moment maps to this complex. We'll see that using a certain extension, forms on the manifold can also be extended to this bi-complex. The main result of the section states that a homotopy moment map corresponds to the primitive of the extension of our multisymplectic form.

We denote the space of all G-invariant forms on M by $\Omega^\bullet(M)^G$ and the complex $\Lambda^{k\geq 1}\mathfrak{g}^* \otimes \Omega^\bullet(M)$ as $C^\bullet_\mathfrak{g}$.
Given any $\sigma \in \Omega^n(M)^G$, we define

$$\begin{aligned} \sigma_k : \bigwedge^k \mathfrak{g} &\to \Omega^{n-k}(M) \\ (x_1, \dots, x_n) &\mapsto \iota(v_{x_1} \wedge \dots \wedge v_{x_n})\sigma \end{aligned}$$

Now, $\tilde{\sigma} := \sum_{k=1}^n (-1)^{k-1}\sigma_k$ can be considered as an element of $C^\bullet_\mathfrak{g}$.

Lemma 4.1.1 ([FLGZ15], Lemma 2.3). *For any $\sigma \in \Omega^n(M)^G$, the extension $\sim: \Omega^n(M)^G \to \Lambda^{\geq 1}\mathfrak{g}^* \otimes \Omega^\bullet(M) : \sigma \mapsto \tilde{\sigma}$ intertwines the differentials, i.e.*

$$d_{tot}\tilde{\sigma} = \widetilde{d\sigma}$$

Proof. We consider Cartan's magic formula, which states $\mathcal{L}_V\Omega = d(\iota_V\Omega) + \iota_V(d\Omega)$ for $V \in \mathfrak{X}(M)$. To prove this lemma, we will need to use a generalized version of this formula, called the 'extended Cartan formula' [Lemma 2.1, [FLGZ15]], which states that for $v_1, \dots, v_k \in \mathfrak{X}(M)$:

$$\begin{aligned} (-1)^k d\iota(v_1 \wedge \dots \wedge v_k)\Omega &= \sum_{1\leq i<j\leq k} (-1)^{i+j}\iota([v_i, v_j], v_1, \dots, \hat{v_i}, \dots, \hat{v_j}, \dots, v_k)\Omega \\ &+ \sum_{i=1}^k (-1)^i \iota(v_1 \wedge \ldots \wedge \hat{v_i} \wedge \ldots \wedge v_k)\mathcal{L}_{v_i}\Omega + \iota(v_1 \wedge \ldots \wedge v_k)d\Omega \end{aligned}$$

Using this, we can deduce that $(-1)^k d\sigma_k = d_\mathfrak{g}\sigma_{k-1} + (d\sigma)_k$ as follows:
Given $x_1, \dots x_k \in \mathfrak{g}$, we have

$$\begin{aligned} (-1)^k d\iota(v_{x_1} \wedge \dots \wedge v_{x_k})\sigma &= \sum_{1\leq i<j\leq k} (-1)^{i+j}\iota([v_{x_i}, v_{x_j}], v_{x_1}, \dots, \hat{v_{x_i}}, \dots, \hat{v_{x_j}}, \dots, v_{x_k})\sigma \\ &+ \sum_{i=1}^k (-1)^i \iota(v_{x_1} \wedge \ldots \wedge \hat{v_{x_i}} \wedge \ldots \wedge v_{x_k})\mathcal{L}_{v_{x_i}}\sigma + \iota(v_{x_1} \wedge \ldots \wedge v_{x_k})d\sigma \end{aligned}$$

As σ is G-invariant, $\mathcal{L}_{v_{x_i}}\sigma = 0$ for all $x_i \in \mathfrak{g}$. Moreover,

$$\begin{aligned} \iota(v_{x_1} \wedge \ldots v_{x_k})d\sigma &= (d\sigma)_k(x_1 \wedge \ldots \wedge x_k) \\ \sum_{1\leq i<j\leq k} (-1)^{i+j}\iota([v_{x_i}, v_{x_j}], v_{x_1}, \dots, \hat{v_{x_i}}, \dots, \hat{v_{x_j}}, \dots, v_{x_k})\sigma &= \sigma_{k-1}(\partial(x_1 \wedge \ldots x_k)) \\ &= d_\mathfrak{g}\sigma_{k-1}(x_1 \wedge \ldots x_k) \end{aligned}$$

Thus, we get

$$(-1)^k d\sigma_k(x_1 \wedge \dots x_k) = d_{\mathfrak{g}}(x_1 \wedge \dots x_k)\sigma_{k-1} + (d\sigma)_k(x_1 \wedge \dots x_k)$$

This proves the claim. From this, we can deduce

$$\begin{aligned} d_{tot}\tilde{\sigma} &= \sum_{k=1}^{n}(-1)^{k-1}(d_{\mathfrak{g}}\sigma_k + (-1)^k d\sigma_k) \\ &= \sum_{k+2}^{n+1}(-1)^k d_{\mathfrak{g}}\sigma_{k-1} - \sum_{k=1}^{n} d\sigma_k \\ &= \sum_{k=2}^{n+1}((-1)^k d_{\mathfrak{g}}\sigma_{k-1} - d\sigma_k) - d\sigma_1 \\ &= \widetilde{d\sigma} \end{aligned}$$

□

So, if $\omega \in \Omega^{n+1}(M)^G$ is closed, then $d_{tot}\tilde{\omega} = \widetilde{d\omega} = 0$, i.e. $\tilde{\omega}$ is closed as well.
If, however, $\tilde{\omega}$ is exact, then the following lemma states that its primitive corresponds to a homotopy moment map of the Lie group action.

Theorem 4.1.2 ([FLGZ15], Proposition 2.5). *Let $\varphi = \varphi_1 + \dots + \varphi_n$ with $\varphi_k \in \Lambda^k\mathfrak{g}^* \otimes \Omega^{n-k}(M)$. Then $d_{tot}\varphi = \tilde{\omega}$ iff for $k = 1, \dots, n$,*

$$f_k := \zeta(k)\varphi_k : \bigwedge^k \mathfrak{g} \to \Omega^{n-k}(M)$$

are components of a homotopy moment map for the action of G on (M, ω).

Proof. The condition $d_{\mathrm{tot}}\varphi = \tilde{\omega}$ expands as:

$$d_{\mathrm{tot}}\varphi = \sum_{k=2}^{n+1} d_g\varphi_{k-1} + \sum_{k=1}^{n}(-1)^k d\varphi_k = \tilde{\omega},$$

which is equivalent to the system of equations:

$$-d\varphi_1 = \omega_1, \tag{4.3}$$

$$d_g\varphi_{k-1} + (-1)^k d\varphi_k = (-1)^{k-1}\omega_k, \quad \text{for } 2 \leq k \leq n, \tag{4.4}$$

$$d_g\varphi_n = (-1)^n\omega_{n+1}. \tag{4.5}$$

Evaluating equation (4.3) on $x \in \mathfrak{g}$, we obtain $d\varphi_1(x) = -\iota_{v_x}\omega$.
For equation (4.4), evaluating on $x_1, \dots, x_k \in \mathfrak{g}$ gives:

$$\sum_{1\leq i<j\leq k} (-1)^{i+j}\varphi_{k-1}([v_{x_i}, v_{x_j}], v_{x_1}, \dots, \hat{v}_{x_i}, \dots, \hat{v}_{x_j}, \dots, v_{x_k}) = -(-1)^k d\varphi_k(v_{x_1}, \dots, v_{x_k}) + (-1)^{k-1}\iota(v_{x_1} \wedge \dots \wedge v_{x_k})\omega.$$

Multiplying this equation by $-\varsigma(k-1) = -(-1)^k\varsigma(k)$, we obtain the desired condition.
Similarly, one sees that equation (4.5) is equivalent to

$$\sum_{1\leq i<j\leq n+1} (-1)^{i+j+1} f_n([x_i, x_j], x_1, \dots, \hat{x}_i, \dots, \hat{x}_j, \dots, x_{n+1}) = \varsigma(n+1)\iota(v_{x_1} \wedge \dots \wedge v_{x_{n+1}})\omega.$$

□

Thus, we can show that the action of a Lie group G on an n-plectic manifold (M,ω) is Hamiltonian by showing that the corresponding extension $\tilde{\omega}$ is exact. Thus, if we have an n-plectic form $\omega \in \Omega^{n+1}(M)$ which has G-invariant primitive σ, then by Lemma 4.1.1, we can conclude that $\tilde{\omega} = \widetilde{d\sigma} = d_{tot}\tilde{\sigma}$. This implies, by Theorem 4.1.2, that the action is Hamiltonian and $\tilde{\sigma}$ corresponds to its homotopy moment map. Thus, we have proved the following:

Corollary 4.1.3. *Let (M,ω) be an n-plectic manifold and G be a Lie group acting on M and preserving ω. If ω has a G-invariant primitive, then the action of G admits a homotopy moment map $(f_k)_k$ given by:*

$$\begin{aligned} f_k : \Lambda^k\mathfrak{g} &\to \Omega^{n-k}(M) \\ x_1 \wedge \ldots \wedge x_k &\mapsto \varsigma(k)\ \iota_{v_{x_1}\wedge\ldots\wedge v_{x_k}}\sigma \end{aligned}$$

Note that the proof of the corollary given in the literature we refer to differs from the one given in this thesis. In the literature, the proof is often given in the setting of the Cartan model. For more details regarding the proof, as well as an extension of the previous Corollary, refer to Section 4.2.

Example 4.1.4. We again consider a multi-cotangent bundle $\Lambda^n T^*M$ with the tautological form $\theta \in \Omega^n(\Lambda^n T^*M)$ as described in Example 3.1.3. We consider a Lie group action $G \curvearrowright M$ given as $\vartheta : G \times M \to M$. We can then extend the action onto $\Lambda^n T^*M$ as

$$\begin{aligned} \tilde{\vartheta}_g : \Lambda^n T^*M &\to \Lambda^n T^*M \\ (p_1,\alpha_1) &\mapsto (p_2,\alpha_2) \end{aligned}$$

with $\alpha_i \in \Lambda^n T^*_{p_i}M$ such that $p_2 = \vartheta_g(p_1)$, $\alpha_1 = \vartheta_g^*\alpha_2$. Note that as $\vartheta_g : M \to M$ is a diffeomorphism for all $g \in G$, we can claim that $\tilde{\vartheta}_g^*\theta = \theta\ \ \forall g \in G$. For a proof of this claim, we refer to Proposition 2.1 in [DS01]. The proof only covers the symplectic case, i.e., only the cotangent bundle T^*M. However, it can be easily extended to the multisymplectic case. Thus, we see that if we have a Lie group action on the base manifold $G \curvearrowright M$, then the tautological form θ would be G-invariant for the extended action on the multi-cotangent bundle $\Lambda^n T^*M$.
Thus, the n-plectic form $d\theta$ has an invariant primitive. Thus, by the previous corollary, we can deduce that the extended G action on $(\Lambda^n T^*M, d\theta)$ admits a homotopy moment map

Corollary 4.1.5. *The canonical action*

$$\mathrm{SO}(n) \curvearrowright \left(\mathbb{R}^n, dx^1 \wedge \cdots \wedge dx^n\right),$$

where $x = (x^i)$ are the standard coordinates on $\mathbb{R}^n$ and $dx^1 \wedge \cdots \wedge dx^n$ is the standard volume form of $\mathbb{R}^n$, admits a homotopy moment map given by (for $k = 1,\ldots,n$):

$$f_k : \Lambda^k\mathfrak{g} \to \Omega^{n-1-k}(\mathbb{R}^n), \quad q \mapsto (-1)^{k-1}\frac{\varsigma(k)}{n}\iota(E \wedge v_q)(dx^1 \wedge \cdots \wedge dx^n),$$

where $E = \sum_i x^i\partial_i$ is the Euler vector field.

Proof. The proof follows from the previous corollary, noting that the standard volume form $\omega = dx^1 \wedge \cdots \wedge dx^n$ admits the SO(n)-invariant form

$$\alpha = \frac{1}{n}\iota_E \left(dx^1 \wedge \cdots \wedge dx^n\right)$$

as a primitive, since $d\alpha = \omega$ and E is SO(n)-invariant. Using the previous corollary with this invariant primitive α gives the expression for f_k:

$$f_k(q) = (-1)^{k-1}\varsigma(k)\iota(v_q)(\alpha) = (-1)^{k-1}\varsigma(k)\iota(v_q)\left(\frac{1}{n}\iota_E\omega\right) = (-1)^{k-1}\frac{\varsigma(k)}{n}\iota(E \wedge v_q)\omega,$$

which proves the claim. □

It can be proven that for a multisymplectic vector space, any linear action of a Lie group that preserves the multisymplectic form would be Hamiltonian. The above Corollary would then pop out as a special case of this. For further details, refer to Example 8.3 in [CFRZ16].

4.2 The Cartan Model

This section focuses on the Cartan model, which is a model used to compute the Equivariant cohomology of a manifold equipped with a group action. We begin this section by introducing the Cartan model and providing the necessary definitions, before turning to the main goal: proving an analogue of Lemma 4.1.2 in the setting of the Cartan model (we'll see, however, that this result can be generalized a bit more). For some basic details on Equivariant Cohomology, refer to Appendix A.
Given a Lie group action $\vartheta : G \times M \to M$, the Cartan model is defined as

$$C_G(M) = (S(\mathfrak{g}^*) \times \Omega^\bullet(M))^G$$

where the degree of $\mathfrak{g}^*$ is 2. Given $\xi \otimes \alpha \in C_G(M)$, with $\xi \in \mathfrak{g}^*$ and $\alpha \in \Omega^\bullet(M)$, a Lie group element $g \in G$ acts on it as

$$g \cdot (\xi \otimes \alpha) = Ad_g^*\xi \otimes \vartheta_g^*\alpha$$

Also, the G in the superscript means that we only consider the G-invariant elements of the complex. We now show in the following proposition that elements of the subspace $(\mathfrak{g}^* \otimes \Omega^{n-1}(M))^G$ can be considered as G-equivariant maps $\mathfrak{g} \to \Omega^{n-1}(M)$.

Proposition 4.2.1 (Prop 6.18 in [AB84])**.** *Let G be a connected Lie group. Given a Lie group action on a manifold $G \curvearrowright M$, consider $\mu \in (\mathfrak{g}^* \otimes \Omega^{n-1}(M))^G$. Then, μ corresponds to an equivariant map $\tilde{\mu} : \mathfrak{g} \to \Omega^{n-1}(M)$.*

Proof. We first fix a basis y_i of $\mathfrak{g}$ and a corresponding dual basis ξ^i of $\mathfrak{g}^*$. Now, if $\tilde{\mu} : \mathfrak{g} \to \Omega^{n-1}(M)$ is the map corresponding to μ, we see that

$$\mu = \sum_i \xi^i \otimes \tilde{\mu}(y_i)$$

We need to show that $\tilde{\mu}$ is equivariant, i.e. for $x, y \in \mathfrak{g}$,

$$\mathcal{L}_{v_x}\tilde{\mu}(y) = \tilde{\mu}([x, y])$$

Consider $\sum_i \xi^i \otimes \tilde{\mu}(y_i)$ to be G-invariant, i.e. we have for all $g \in G$,

$$g \cdot \left(\sum_i \xi^i \otimes \tilde{\mu}(y_i) \right) = \sum_i \xi^i \otimes \tilde{\mu}(y_i)$$
$$\implies \sum_i Ad_g^* \xi^i \otimes \vartheta_g^* \tilde{\mu}(y_i) = \sum_i \xi^i \otimes \tilde{\mu}(y_i)$$

Considering $g = \exp(tx)$ for some $x \in \mathfrak{g}$ and taking the derivative with respect to t, we get

$$\sum_i \left(ad_x^* \xi^i \otimes \tilde{\mu}(y_i) + \xi^i \otimes \mathcal{L}_{v_x} \tilde{\mu}(y_i) \right) = 0, \text{i.e.}$$
$$\sum_i \xi^i \otimes \mathcal{L}_{v_x} \tilde{\mu}(y_i) = \sum_i ad_{-x}^* \xi^i \otimes \tilde{\mu}(y_i)$$

Note that the left-hand side corresponds to a map $\mathfrak{g} \to \Omega^{n-1}(M) : y \mapsto \mathcal{L}_{v_x} \tilde{\mu}(y)$. To see how the map corresponding to the right-hand term works, we pair it with a vector $y \in \mathfrak{g}$:

$$\begin{aligned} \left(\sum_i ad_{-x}^* \xi^i \otimes \tilde{\mu}(y_i) \right)(y) &= \sum_i ad_{-x}^* \xi^i(y) \cdot \tilde{\mu}(y_i) \\ &= \sum_i \xi^i(ad_x y) \cdot \tilde{\mu}(y_i) \\ &= \sum_i \xi^i([x,y]) \cdot \tilde{\mu}(y_i) \\ &= \tilde{\mu}\left(\sum_i \xi^i([x,y]) \cdot y_i \right) = \tilde{\mu}([x,y]) \end{aligned}$$

In the final equality, we use the fact that for any $x \in \mathfrak{g}$:

$$\sum_i \xi^i(x) \cdot y_i = x$$

Thus, we have that the right hand side corresponds to a map $\mathfrak{g} \to \Omega^{n-1}(M) : y \mapsto \tilde{\mu}([x,y])$. This implies that for all $x, y \in \mathfrak{g}$, we have that

$$\mathcal{L}_{v_x} \tilde{\mu}(y) = \tilde{\mu}([x,y])$$

□

Moreover, the Cartan complex is equipped with the following differential:

$$d_G = 1 \otimes d + \sum_i \xi^i \otimes \iota_{v_{x_i}}$$

where we fix a basis $\{x_i\}_i$ of $\mathfrak{g}$, and we denote corresponding basis of $\mathfrak{g}^*$ as $\{\xi^i\}_i$.

Lemma 4.2.2. *d_G is a differential on $C_G(M)$.*

Proof. It suffices to show that it squares to zero on its generators.
Considering a generator $\alpha \in (\mathfrak{g}^* \times \Omega^\bullet(M))^G \subseteq C_G(M)$ as a polynomial $\alpha : \mathfrak{g} \to \Omega^\bullet(M)$, the differential d_G acts as

$$(d_G\alpha)(x) = d(\alpha(x)) + \iota_{v_x}(\alpha(x)) \quad \forall x \in \mathfrak{g}$$

So we have that

$$\begin{aligned} d_G(d_G\alpha)(x) &= d(d_G\alpha(x)) + \iota_{v_x}(d_G\alpha(x)) \\ &= d(d\alpha(x)) + d\iota_{v_x}(\alpha(x)) + \iota_{v_x}d(\alpha(x)) + \iota_{v_x}\iota_{v_x}(\alpha(x)) \\ &= d\iota_{v_x}(\alpha(x)) + \iota_{v_x}d(\alpha(x)) \\ &= \mathcal{L}_{v_x}\alpha(x) = 0 \end{aligned}$$

□

For any non-negative integer k,

$$C_G^k(M) = \bigoplus_{j=1}^{\lfloor \frac{k}{2} \rfloor} (S^j(\mathfrak{g}^*) \otimes \Omega^{k-2j}(M))$$

is set of all elements of degree k in $C_G(M)$. Also, given $\alpha \in C_G^k(M)$, denote by α_j its projection in $S^j(\mathfrak{g}^*) \otimes \Omega^{k-2j}(M)$. Thus, we have that $\alpha = \sum_{i=1}^{\lfloor \frac{k}{2} \rfloor} \alpha_i$.

Definition 4.2.3. Given an invariant closed differential form $\omega \in \Omega^k(M)$, an **extension** is defined to be a cocycle $\alpha \in C_G^k(M)$ such that

$$\alpha_0 = \omega$$

Definition 4.2.4. Given an invariant closed differential form $\omega \in \Omega^k(M)$, a **j-step extension** is an extension of the form

$$\alpha = \alpha_0 + \alpha_1 + \ldots + \alpha_j$$

Thus, we see that a 1-step extension is $\mu \in (\mathfrak{g}^* \otimes \Omega^{k-2}(M))^G$ such that $\omega + \mu$ is a cocycle in $C_G(M)$. We first see how in symplectic geometry, a 1-step extension is the same as a moment map.

Proposition 4.2.5. *Consider the action of a Lie group on a symplectic manifold $G \curvearrowright (M, \omega)$. A 1-step extension of the symplectic form ω is the same as a moment map.*

Proof. Let $\mu \in (\mathfrak{g}^* \otimes C^\infty(M))^G$ be a 1-step extension of the symplectic form ω. Thus, by Proposition 4.2.1, we can consider $\mu : \mathfrak{g} \to C^\infty(M)$ as an equivariant map such that $d_G(\omega + \mu) = 0$.
Given $x \in \mathfrak{g}$,

$$\begin{aligned} d_G\mu(x) &= d\mu(x) - \iota_{v_x}\mu(x) \\ &= d\mu(x) - \mu([x, x]) \\ &= d\mu(x) \end{aligned}$$

We now fix a basis y_i of $\mathfrak{g}$ and a dual basis ξ^i of $\mathfrak{g}^*$. Now,

$$\begin{aligned} d_G\omega &= d\omega + \sum_i \xi^i \otimes \iota_{v_{y_i}}\omega \\ &= \sum_i \xi^i \otimes \iota_{v_{y_i}}\omega \end{aligned}$$

Thus, for $x \in \mathfrak{g}$, $d_G\omega(x) = \iota_{v_x}\omega$.
Since $d_G\mu = -d_G\omega$, we get that

$$d\mu(x) = -\iota_{v_x}\omega$$

Thus, μ is a moment map for the action of G.
Conversely, consider that the G-action admits a moment map $\mu : \mathfrak{g} \to C^\infty(M)$. Since, μ is equivariant, we have that $\mu \in (S(\mathfrak{g}^*) \otimes \Omega^\bullet(M))^G$.
Following a similar argument, we have that for all $x \in \mathfrak{g}$,

$$\begin{aligned} d_G\mu(x) &= d\mu(x) \\ &= -\iota_{v_x}\omega \qquad \text{(as } \mu \text{ is a moment map)} \\ &= -d_G\omega(x) \end{aligned}$$

Thus, $d_G(\omega + \mu) = 0$, i.e. μ is a 1-step extension of ω. □

While this exact correspondence doesn't generalize to the multisymplectic case, it can be proven that a 1-step extension does correspond to a homotopy moment map for a Lie group action on a multisymplectic manifold. We provide an alternate proof to one presented in the source cited for this Proposition.

Proposition 4.2.6 (Proposition 4.4 in [FLGZ15]). *Let (M, ω) be an n-plectic manifold equipped with a G-action such that $\omega \in \Omega^{n+1}(M)^G$ admits a 1-step extension, then there exists a homotopy moment map for the action of G on (M, ω) given by $\{f_k\}_k$, where*

$$\begin{aligned} f_k : \bigwedge^k \mathfrak{g} &\to \Omega^{n-1}(M) \\ x_1 \wedge \cdots \wedge x_k &\mapsto \varsigma(k+1)\iota_{v_{x_2} \wedge \cdots \wedge v_{x_k}}\mu(x_1) \end{aligned}$$

Proof. We first fix a basis y_i of $\mathfrak{g}$ and the corresponding dual basis ξ^i of $\mathfrak{g}^*$. Now, note the equivariant map $\mu : \mathfrak{g} \to \Omega^{n-1}(M)$ can be written as

$$\mu = \sum_i \xi^i \otimes \mu(y_i)$$

Thus we have that

$$d_G\mu = \sum_i \xi^i \otimes d\mu(y_i) + \sum_k \sum_i \xi^k.\xi^i \otimes \iota_{v_{y_k}}\mu(y_i)$$

Also, we have that

$$\begin{aligned} d_G\omega &= d\omega + \sum_i \xi^i \otimes \iota_{v_{y_i}}\omega \\ &= \sum_i \xi^i \otimes \iota_{v_{y_i}}\omega \end{aligned}$$

as ω is closed. Now, since $d_G(\omega + \mu) = 0$, we get that

$$\begin{aligned} \sum_i \xi^i \otimes (\iota_{v_{y_i}}\omega + d\mu(y_i)) = 0 \\ \implies \iota_{v_{y_i}}\omega = -d\mu(y_i) \;\text{ for every basis vector } y_i \\ \implies \iota_{v_x}\omega = -d\mu(x) \;\; \forall x \in \mathfrak{g} \end{aligned}$$

Also,

$$\sum_k \sum_i \xi^k.\xi^i \otimes \iota_{v_{y_k}}\mu(y_i) = 0$$

Note this term can be a viewed as a map

$$\begin{gathered} \bigwedge^2 \mathfrak{g} \to \Omega^{n-2}(M) \\ x_1 \wedge x_2 \mapsto \iota_{v_{x_1}}\mu(x_2) \end{gathered}$$

This implies that

$$\iota_{v_{x_1}}\mu(x_2) = -\iota_{v_{x_2}}\mu(x_1)$$

Now, we define

$$\begin{gathered} \mu_k : \bigwedge^k \mathfrak{g} \to \Omega^{n-k}(M) \\ x_1 \wedge \cdots \wedge x_k \mapsto \iota_{v_{x_2} \wedge \cdots \wedge v_{x_k}}\mu(x_1) \end{gathered}$$

where $\mu_1 := \mu$. Then, we claim that $f_k := \varsigma(k+1)\mu_k$ form the components of a homotopy moment map, i.e., by Definition 4.0.1, they satisfy the following identity for all $1 \leq k \leq n,\ p \in \bigwedge^k \mathfrak{g}$:

$$-f_{k-1}(\partial p) = df_k(p) + \varsigma(k)\iota_{v_p}\omega \tag{4.6}$$

Given $x_1, \ldots, x_k \in \mathfrak{g}$, let $p = x_1 \wedge \cdots \wedge x_k \in \bigwedge^k \mathfrak{g}^*$ and $p' = x_2 \wedge \cdots \wedge x_k \in \bigwedge^{k-1} \mathfrak{g}^*$

$$\begin{aligned} df_k(p) &= \varsigma(k+1)d(\iota_{v_{p'}}\mu(x_1)) \\ &= (-1)^{k-1}\varsigma(k)d(\iota_{v_{p'}}\mu(x_1)) \\ &= \varsigma(k)(\iota_{v_{p'}}d\mu(x_1) + \iota_{v_{\partial p'}}\mu(x_1) + \sum_{m=2}^{k}(-1)^{m-1}\iota_{v_{x_2}\wedge\cdots\wedge \hat{v_{x_m}}\wedge\cdots\wedge v_{x_k}}\mathcal{L}_{v_{x_m}}\mu(x_1)) \end{aligned}$$

Here, we used another variant of the multivector version of Cartan's magic formula, which states that

$$(-1)^m d\iota_{x_1\wedge\cdots\wedge x_m} = \iota_{x_1\wedge\cdots\wedge x_m}d + \iota_{\partial(x_1\wedge\cdots\wedge x_m)} + \sum_k (-1)^k \iota_{x_1\wedge\cdots\wedge\hat{x_k}\wedge\cdots\wedge x_m}\mathcal{L}_{v_{x_k}}$$

For the proof of the formula, the interested reader can refer Lemma 3.4 in [MS12]. Also, note that since μ is an equivariant map, we have that for $x, y \in \mathfrak{g}$, $\mathcal{L}_{v_x}\mu(y) = \mu([x, y])$

$$
\begin{aligned}
f_{k-1}(\partial p) &= \varsigma(k)\mu_{k-1}\left(\sum_{1\le i<j\le k}(-1)^{i+j}[x_i,x_j]\wedge x_1\wedge\cdots\wedge\hat{x_i}\wedge\cdots\wedge\hat{x_j}\wedge\cdots\wedge x_k\right)\\
&= \varsigma(k)\mu_{k-1}\left(\sum_{2\le m\le k}(-1)^{m+1}[x_1,x_m]\wedge x_2\wedge\cdots\wedge\hat{x_m}\wedge\cdots\wedge x_k\right)\\
&\quad + \varsigma(k)\mu_{k-1}\left(\sum_{2\le i<j\le k}(-1)^{i+j}[x_i,x_j]\wedge x_1\wedge\cdots\wedge\hat{x_i}\wedge\cdots\wedge\hat{x_j}\wedge\cdots\wedge x_k\right)\\
&= -\varsigma(k)\mu_{k-1}\left(\sum_{2\le m\le k}(-1)^{m}[x_1,x_m]\wedge x_2\wedge\cdots\wedge\hat{x_m}\wedge\cdots\wedge x_k\right)\\
&\quad - \varsigma(k)\mu_{k-1}\left(\sum_{2\le i<j\le k}(-1)^{i+j}x_1\wedge[x_i,x_j]\wedge\cdots\wedge\hat{x_i}\wedge\cdots\wedge\hat{x_j}\wedge\cdots\wedge x_k\right)\\
&= -\varsigma(k)\left(\sum_{2\le m\le k}(-1)^{m}\ \iota_{v_{x_2}\wedge\cdots\wedge\hat{v_{x_m}}\wedge\cdots v_{x_k}}\mu([x_1,x_m]) + \iota_{v_{\partial p'}}\mu(x_1)\right)\\
&= -\varsigma(k)\left(\sum_{2\le m\le k}(-1)^{m-1}\ \iota_{v_{x_2}\wedge\cdots\wedge\hat{v_{x_m}}\wedge\cdots v_{x_k}}\mu([x_m,x_1]) + \iota_{v_{\partial p'}}\mu(x_1)\right)
\end{aligned}
$$

Using $\mathcal{L}_{v_x}\mu(y)=\mu([x,y])$, we get

$$-f_{k-1}(\partial p) = df_k(p) - \varsigma(k)\iota_{v_{x_2}\wedge\cdots\wedge v_{x_k}}d\mu(x_1)$$

We know that

$$
\begin{aligned}
\varsigma(k)\iota_{v_p}\omega &= \varsigma(k)\iota_{v_{x_1}\wedge\cdots\wedge v_{x_k}}\omega\\
&= \varsigma(k)\iota_{v_{x_2}\wedge\cdots\wedge v_{x_k}}\iota_{v_{x_1}}\omega\\
&= -\varsigma(k)\iota_{v_{x_2}\wedge\cdots\wedge v_{x_k}}d\mu(x_1)
\end{aligned}
$$

Thus, we get that for $1\le k\le n$,

$$-f_{k-1}(\partial p) = df_k(p) + \varsigma(k)\iota_{v_p}\omega$$

Since $(f_k)_k$ satisfies eq (4.6), it is a homotopy moment map for (M,ω). □

Now, we can use the above proposition to give an alternative proof of Corollary 4.1.3. Note that if the multisymplectic form ω has an invariant primitive α, then $\tilde{\alpha}$ is the 1-step extension of ω, where

$$
\begin{aligned}
\tilde{\alpha} : \mathfrak{g} &\to \Omega^{n-1}(M)\\
x &\mapsto \iota_{v_x}\alpha
\end{aligned}
$$

Thus

$$
\begin{aligned}
d_G\tilde{\alpha}(x) &= d\alpha(x) + \iota_{v_x}\tilde{\alpha}(x)\\
&= d\iota_{v_x}\alpha + \iota_{v_x}\iota_{v_x}\alpha\\
&= d\iota_{v_x}\alpha\\
&= \mathcal{L}_{v_x}\alpha - \iota_{v_x}d\alpha\\
&= -\iota_{v_x}\omega
\end{aligned}
$$

In the final equality, we use $\mathcal{L}_{v_x}\alpha = 0$ as α is G-invariant. Thus, we have that

$$d_G\tilde{\alpha} = -\sum_i \xi^i \otimes \iota_{v_{y_i}}\omega = -d_G\omega$$

i.e., $d_G(\omega + \tilde{\alpha}) = 0$.
It is possible to extend Proposition 4.2.6 as follows: If for an n-plectic manifold (M, ω), ω admits any arbitrary extension in $C_G(M)$, then the G-action admits a homotopy moment map. For a proof of this statement, refer to Theorem 6.8 in [CFRZ16].

4.3 Compact Lie group actions

Given the action of a Lie group $\vartheta : G \times M \to M$, we denote by $\Omega^\bullet(M, \vartheta)$ the complex of G-invariant differential forms on M.

In this section, we shall deal with the specific case of a compact Lie group acting on a multisymplectic manifold. We shall mostly refer to [MR20] for the entirety of the section. We'll see that the condition for the existence of homotopy moment maps simplifies a little when considering the specific case of compact Lie groups. We show that the obstruction class for the action of the compact Lie group is given by $[\vartheta^*\omega - \pi^*\omega] \in H^n(G \times M)$ and apply this result to case of spheres.

Lemma 4.3.1. *Let $\vartheta : G \times M \to M$ be a right Lie group action. We denote by*

$$\begin{aligned}(r \times \mathrm{id}) : G \times (G \times M) &\to (G \times M)\\ (h, (g, m)) &\mapsto (gh, m)\end{aligned}$$

the right multiplication action on the second factor. Then the complex $\Omega^\bullet(G \times M, r \times \mathrm{id})$ is naturally isomorphic to $C^\bullet_{\mathfrak{g}} \oplus (\Lambda^0\mathfrak{g}^* \otimes \Omega^\bullet(M))$.

Proof. We have a natural map:

$$\Lambda^\bullet\mathfrak{g}^* \otimes \Omega^\bullet(M) \to \Omega^\bullet(G, r) \otimes \Omega^\bullet(M) \to \Omega^\bullet(G \times M, r \times \mathrm{id}).$$

The first map identifies $\Lambda^\bullet\mathfrak{g}^*$ with right-invariant forms on G, while the second uses the wedge product:

$$\alpha \otimes \beta \mapsto \pi_1^*\alpha \wedge \pi_2^*\beta,$$

where π_i are projections onto the factors. As the complexes involved are graded vector spaces, we need to take the degree of the element into consideration and use the Koszul sign convention. So, the map extends to:

$$\alpha \otimes \beta \mapsto (-1)^{|\alpha|}\pi_1^*\alpha \wedge \pi_2^*\beta.$$

An inverse is given by restricting to $\{e\} \times M$, which identifies a form in $\Omega^\bullet(G \times M, r \times \mathrm{id})$ with an element of $\Lambda^\bullet\mathfrak{g}^* \otimes \Omega^\bullet(M)$. Note that

$$\Lambda^\bullet\mathfrak{g}^* \otimes \Omega^\bullet(M) \cong (\Lambda^{\geq 1}\mathfrak{g}^* \otimes \Omega^\bullet(M)) \oplus (\Lambda^0\mathfrak{g}^* \otimes \Omega^\bullet(M)) = C^\bullet_{\mathfrak{g}} \oplus (\Lambda^0\mathfrak{g}^* \otimes \Omega^\bullet(M))$$

Thus, we get an isomorphism of complexes and cohomology. □

Proposition 4.3.2. *Assume that G preserves a multisymplectic form ω. Let v be the infinitesimal action induced by G. Then the cocycle $\tilde{\omega} \in C^{n+1}_{\mathfrak{g}}$ (defined in Section 4.1) is given by:*

$$\tilde{\omega} = \vartheta^*\omega - \pi^*\omega,$$

where $\pi : G \times M \to M$ is the projection onto the second factor.

Proof. The map $\vartheta : G \times M \to M$ is equivariant with respect to $(r \times \mathrm{id})$ on $G \times M$ and ϑ on M. Therefore, ϑ^* restricts to a well-defined map on invariant subcomplexes. For $\xi_1, \ldots, \xi_k \in \mathfrak{g}$ and $X_1, \ldots, X_{n+1-k} \in T_pM$, one computes:

$$\begin{aligned}\vartheta^*\omega(\xi_1, \ldots, \xi_k, X_1, \ldots, X_{n+1-k}) &= \omega(v(\xi_1), \ldots, v(\xi_k), X_1, \ldots, X_{n+1-k}) \\ &= (\iota^k_{\mathfrak{g}}\omega)(\xi_1, \ldots, \xi_k)(X_1, \ldots, X_{n+1-k})\end{aligned}$$

Here, for $0 < k \leq n+1$,

$$\begin{aligned}\iota^k_{\mathfrak{g}}\omega : \Lambda^k\mathfrak{g} &\to \Omega^{n-k}(M) \\ x_1 \wedge \cdots \wedge x_n &\mapsto \iota(v_{x_1} \wedge \cdots \wedge v_{x_k})\omega\end{aligned}$$

For $k = 0$ we have $\vartheta^*\omega = \omega$, and thus:

$$\vartheta^*\omega = \sum_{k=0}^{n+1}(\iota^k_{\mathfrak{g}}\omega),$$

so that:

$$\tilde{\omega} = \vartheta^*\omega - \pi^*\omega = \sum_{k=1}^{n+1}(-1)^{k-1}\iota^k_{\mathfrak{g}}\omega,$$

as required. □

Note that the above proposition can also be used to show that a G-invariant primitive α of ω will induce a homotopy moment map as $(\vartheta^*\alpha - \pi^*\alpha) \in \Omega^\bullet(G \times M, r \times \mathrm{id})$ would be a potential of $\tilde{\omega}$. In fact, since we are mostly dealing with compact Lie groups in this section, we don't even have to worry about invariance: this is because given any form on the manifold, we can *'average'* it over the Lie group, which turns it into an invariant form. This trick is called 'averaging over compact Lie groups' as it involves an integral over the Lie group. For more details, the interested reader is directed towards Theorem 1.28 in [FOT08].

Before moving on, we point out that via this averaging trick, the inclusion $\iota : H^\bullet(M, \vartheta) \to H^\bullet(M)$ turns into an isomorphism for a compact and connected Lie group action ϑ. Here, $H^\bullet(M, \vartheta)$ is the *invariant deRham cohomology* of M, which is just the deRham cohomology of the complex of invariant forms: $\Omega^\bullet(M, \vartheta)$. Given $\alpha \in \Omega^\bullet(M, \vartheta)$, $g \in G$, we can find a curve $\gamma : [0, 1] \to G$ connecting g to e. Then the smooth family of diffeomorphisms $(\vartheta_{\gamma(t)})_{t\in[0,1]} : M \to M$ would be an isotopy to identity. Thus, the inverse map

$$\begin{aligned}\pi : \Omega^\bullet(M) &\to \Omega^\bullet(M, \vartheta) \\ \theta &\mapsto \int_G \theta\end{aligned}$$

where $\int_G \theta$ is the averaged (invariant) form on M, is an isomorphism on the deRham cohomology, i.e.

$$\left[\int_G \theta\right] = [\theta]$$

Corollary 4.3.3. *Let $\vartheta : G \times M \to M$ be a compact Lie group acting on a multisymplectic manifold, preserving the multisymplectic form ω. A homotopy moment map exists if and only if $[\vartheta^*\omega - \pi^*\omega] = 0$ in $H^{n+1}(G \times M)$.*

Proof. From Proposition 4.3.2 and lemma 4.1.2, the obstruction to the existence of a homotopy moment map is the class of

$$\tilde{\omega} = \vartheta^*\omega - \pi^*\omega \in \Omega^{n+1}(G \times M).$$

We trace how this class propagates through cochain complexes and cohomologies via the following sequence :

$$\begin{array}{ccc}
\Omega^\bullet(M,\vartheta) & H_{dR}(M) & [\omega] \\
\big\downarrow{\scriptstyle \vartheta^*-\pi^*} & \big\downarrow{\scriptstyle \vartheta^*-\pi^*} & \big\downarrow \\
\Omega^\bullet(G \times M, r \times id) & H_{dR}(G \times M) & [\vartheta^*\omega - \pi^*\omega] \\
\big\downarrow{\scriptstyle \cong} & \big\downarrow{\scriptstyle (Kunneth)\ \cong} & \big\downarrow \\
\Omega^\bullet(G,r) \otimes \Omega^\bullet(M) & H_{dR}(G) \otimes H_{dR}(M) & \\
\big\downarrow{\scriptstyle \cong} & \big\downarrow{\scriptstyle \cong} & \\
\Lambda^\bullet\mathfrak{g}^* \otimes \Omega^\bullet(M) & H_{CE}(\mathfrak{g}) \otimes H_{dR}(M) & \\
\big\downarrow{\scriptstyle \cong} & \big\downarrow{\scriptstyle \cong} & \big\downarrow \\
C^\bullet_{\mathfrak{g}} \oplus (\mathbb{R} \otimes \Omega^\bullet(M)) & H(C^\bullet_{\mathfrak{g}}) \oplus H_{dR}(M) & [\tilde{\omega}]
\end{array}$$

Here, in the first and the third line, we use the fact that since G is a compact Lie group, the invariant deRham cohomology groups are isomorphic to the deRham cohomology groups. In the third map, we use the fact that for a compact, connected Lie group G, $H_{CE}(\mathfrak{g}) \cong H_{dR}(G)$, and in the final map on the cohomology groups, we have

$$\begin{aligned}
H_{CE}(\mathfrak{g}) \otimes H_{dR}(M) &\cong \bigoplus_{k=0}^{\infty} H^k_{CE}(\mathfrak{g}) \otimes \bigoplus_{l=0}^{\infty} H^l_{dR}(M) \\
&= \left(\bigoplus_{k=1}^{\infty} H^k_{CE}(\mathfrak{g}) \otimes \bigoplus_{l=0}^{\infty} H^l_{dR}(M) \right) \oplus \left(H^0_{CE}(\mathfrak{g}) \oplus \bigoplus_{k=0}^{\infty} H^k_{dR}(M) \right) \\
&= H(C^\bullet_{\mathfrak{g}}) \oplus (\mathbb{R} \otimes H_{dR}(M)) \cong H(C^\bullet_{\mathfrak{g}}) \oplus H_{dR}(M)
\end{aligned}$$

The vanishing of $[\tilde{\omega}]$ in any row implies the existence of a homotopy moment map. □

Case: $M = S^n$

In this subsection, we consider the specific case of spheres equipped with their volume form. We build towards the main theorem of [MR20], which states the following:

Proposition 4.3.4 (Main Theorem, [MR20]). *Given a compact Lie group G acting multisymplectically (i.e., action preserves the multisymplectic form) on the n-dimensional sphere S^n, the action admits a homotopy moment map if and only if the action is non-transitive or n is even.*

We first prove the existence of homotopy moment map for the case of non-transitive compact Lie actions.

Lemma 4.3.5. *Let $\vartheta : G \times S^n \to S^n$ be a compact Lie group acting multisymplectically on S^n equipped with the standard volume form $\omega \in \Omega^n(S^n)$. Let $p \in S^n$ be any point. Then a homotopy moment map exists if and only if $\vartheta_p^*[\omega] = 0$ in $H^n(G)$, where $\vartheta_p : G \to S^n$ is the orbit map $g \mapsto gp$.*

Proof. By Corollary 4.3.3, a homotopy moment map exists if and only if $[\vartheta^*\omega - \pi^*\omega] = 0$ in $H^n(G \times S^n)$. Consider the inclusion $i : G \to G \times S^n$, $g \mapsto (g, p)$. Then

$$i^*(\vartheta^*\omega - \pi^*\omega) = \vartheta_p^*\omega - (\pi \circ i)^*(\omega) = \vartheta_p^*\omega,$$

since $\pi \circ i$ is constant and pulls back to 0. So, $[\vartheta^*\omega - \pi^*\omega\ ] = 0$ implies $\vartheta_p^*[\omega] = 0$. Conversely, by the Kunneth formula, we have

$$H^n(G \times S^n) = \bigoplus_{i=0}^{n} H^i(G) \otimes H^{n-i}(S^n)$$

We know that the only non-trivial cohomology groups $H^k(S^n)$ are for $k = 0, n$. Thus,

$$H^n(G \times S^n) \cong H^n(G) \otimes H^0(S^n) \oplus H^0(G) \otimes H^n(S^n).$$

The obstruction class $[\tilde{\omega}] \in H(C_{\mathfrak{g}}^\bullet) \subset H(C_{\mathfrak{g}}^\bullet) \oplus H_{dR}(S^n)$. This subspace maps to $\bigoplus_{k=1}^{\infty} H_{dR}^k(G) \otimes \bigoplus_{l=0}^{\infty} H_{dR}^l(S^n) = H_{dR}^n(G) \otimes H_{dR}^0(S^n)$ under the following isomorphism from Corollary 4.3.3:

$$H(C_{\mathfrak{g}}^\bullet) \oplus H_{dR}(S^n) \cong H_{dR}(G) \otimes H_{dR}(S^n)$$

Thus, we can see that the obstruction class $[\tilde{\omega}] = [\vartheta^*\omega - \pi^*\omega]$ only has components in $H^n(G) \otimes H^0(S^n)$. The vanishing of $i^*[\tilde{\omega}] = \vartheta_p^*[\omega]$ implies $[\vartheta^*\omega - \pi^*\omega] = 0$. □

Note that the above proposition doesn't just hold for spheres, it holds for all n-plectic manifolds such that

$$\bigoplus_{k=1}^{n} H^k(G) \otimes H^{n-k}(M) = 0$$

Proposition 4.3.6. *Let G be a compact Lie group acting non-transitively on S^n and preserving the standard volume form. Then G admits a homotopy moment map.*

Proof. If G acts non-transitively, then some orbit $\mathcal{O} \subset S^n$ has $\dim \mathcal{O} < n$. Let $p \in \mathcal{O}$. Then $\vartheta_p^*[\omega] = 0$ since $\omega|_{\mathcal{O}} = 0$ for dimensional reasons. By Lemma 4.3.5, this implies that the action admits a homotopy moment map. □

Constructing explicit homotopy moment maps

In what follows, we shall try to construct such an explicit homotopy moment map for the action of $SO(n)$ on S^n. For that, we'll require the fact that homotopy moment maps are well-behaved under restriction to a Lie subgroup or an invariant submanifold. All of these results are stated in the following proposition:

Proposition 4.3.7. *Let $G \curvearrowright (M,\omega)$ be a multisymplectic group action. Suppose there exist:*

- *a multisymplectic manifold (N,η) with a G-invariant embedding $j : M \hookrightarrow N$,*
- *a Lie group $H \supset G$,*
- *a multisymplectic action $H \curvearrowright (N,\eta)$ with a homotopy moment map $s : \mathfrak{h} \to L^\infty(N,\eta)$,*
- *a cycle $p \in Z_k(\mathfrak{h}) \subset \Lambda^k\mathfrak{h}$ such that $G \subset H_p$ and $\omega = j^*\iota(v_p)\eta$, where H_p is the corresponding isotropy group for the adjoint action of H on $\Lambda^k\mathfrak{g}$*

then the action $G \curvearrowright (M,\omega)$ admits a homotopy moment map, given by:

$$f_i : \Lambda^i\mathfrak{g} \to \Omega^{n-k-i}(M), \quad q \mapsto (-1)^k j^*\iota(v_q)(s_k(p)),$$

for $i = 1,\dots,n-k$.

We refer to section 2.1 of [MR20] for the proof.

Now, consider the $SO(n+1)$ invariant embedding $j : S^n \hookrightarrow \mathbb{R}^{n+1}$. Considering the standard coordinates on $\mathbb{R}^{n+1} : x = (x_0,\dots,x_n)$, we equip it with the volume form $dx^{0\dots n} := dx^0\wedge\dots\wedge dx^n$. Then, the volume form on the unit sphere can be expressed as $\omega = j^*\iota_E dx^{0,\dots n}$, where E is the Euler vector field. The Euler vector field can seen as the fundamental vector field of the action of $\mathbb{R}$ on $\mathbb{R}^{n+1}$:

$$\vartheta : \mathbb{R}\times\mathbb{R}^{n+1} \to \mathbb{R}^{n+1} : (\lambda, x) \mapsto e^\lambda x$$

It can also be seen as the infinitesimal action of the identity matrix $\mathbb{1}_{n+1} \in \mathfrak{gl}(n,\mathbb{R}^{n+1})$.

We define $H := SO(n)\times\mathbb{R}$. The corresponding Lie algebra is generated by

$$\mathfrak{h} = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix} \middle| \; a \in \mathfrak{so}(n) \right\} \;\oplus\; \langle \mathbb{1}_{n+1}\rangle \cong \mathfrak{so}(n)\oplus\mathbb{R}$$

The group H acts on $\mathbb{R}^{n+1}$ via the infinitesimal action:

$$\begin{aligned} v : \mathfrak{h} &\to \mathfrak{X}(\mathbb{R}^{n+1}) \\ A &\mapsto \sum_{i,j} [A]_{ij}\; x^j\partial_i \end{aligned}$$

We use the following notation:

$$r = \sqrt{(x^1)^2 + \cdots (x^n)^2} \qquad\qquad R = \sqrt{(x^0)^2 + r^2}$$

Lemma 4.3.8. *The differential form*

$$\eta = \Theta\, dx^0\wedge\cdots\wedge dx^n \in \Omega^{n+1}(\mathbb{R}^{n+1}\setminus\{0\}), \quad \textit{with } \Theta = \frac{1}{R^{n+1}},$$

is multisymplectic on $N = \mathbb{R}^{n+1}\setminus\{0\}$, invariant under the action of $H = \mathrm{SO}(n)\times\mathbb{R}$, and restricts to the standard volume form on the unit sphere $S^n \subset \mathbb{R}^{n+1}$.

The function $\Theta \in C^\infty(\mathbb{R}^{n+1})$ is precisely the scaling factor that makes the Euclidean volume invariant under the action the extended group $SO(n+1)$. The problem of finding an explicit homotopy moment map for the action of H can be resolved by finding an H-invariant primitive of η, which we do in the following lemma:

Lemma 4.3.9 ([MR20], Lemma 2.13)**.** *The differential $(n+1)$-form $\eta = \Theta\, dx^0 \wedge \cdots \wedge dx^n$ on $\mathbb{R}^{n+1} \setminus \{0\}$ admits an H-invariant potential $\beta \in \Omega^n(\mathbb{R}^{n+1} \setminus \{0\})$, given by:*

$$\beta = \widehat{\varphi}(x^0, r)\, x^0\, dx^1 \wedge \cdots \wedge dx^n,$$

where the smooth function $\widehat{\varphi}$ depends only on the cylindrical coordinates (x^0, r) and is defined by:

$$\widehat{\varphi}(x^0, r) = \begin{cases} \dfrac{1}{((x^0)^2 + r^2)^{\frac{n+1}{2}}} \left(x^0(n+1) - r \arctan\left(\dfrac{x^0}{r}\right)\right), & r \neq 0, \\ (n+1) \cdot |x^0|^{-n}, & r = 0. \end{cases}$$

Proposition 4.3.10. *A homotopy moment map for the action $SO(n) \curvearrowright (S^n, \omega)$ for $n \geq 2)$ is given by*

$$\begin{aligned} f_i : \Lambda^i \mathfrak{so}(n) &\to \Omega^{n-1-i}(S^n) \\ q &\mapsto -j^* \iota(v_q)(\iota_E \beta) \end{aligned}$$

where β is the primitive from the previous lemma.

Proof. We use Corollary 4.3.7 considering:

$$\begin{aligned} &(N, \eta) = (\mathbb{R}^{n+1},\ \Theta dx^{0\ldots n}) && (M, \omega) = (S^n,\ j^* \iota_E dx^{0\ldots n}) \\ &p = \mathbb{1}_{n+1} \in Z_1(\mathfrak{h}) && H = SO(n) \times \mathbb{R} = H_E \supset SO(n) \end{aligned}$$

and noting that Corollary 4.1.3 gives an explicit homotopy moment map for the action of H as η admits an invariant primitive. □

As mentioned at the beginning of the section, even transitive actions of compact Lie groups admit a homotopy moment map, provided that the dimension of the sphere is even.

Theorem 4.3.11 (Theorem 3.1 in [MR20])**.** *Given a compact Lie group G acting multi-symplectically, transitively, and effectively on S^n equipped with its volume form, the action admits a homotopy moment map if and only if n is even*

Thus, if the action on S^n is non-transitive, then Proposition 4.3.6 implies that it admits a homotopy moment map. If the action is transitive, then Proposition 4.3.11 implies that a homotopy moment exists if and only if n is even. Combining the two statements, we get Proposition 4.3.4.

We shall not prove Theorem 4.3.11 in its full generality, however, we do prove that it holds for the special case of $SO(n+1)$.

Proposition 4.3.12. *Let ω be the volume form of S^{2n}. Let N be the north pole and define $\vartheta_N : \mathrm{SO}(2n+1) \to S^{2n}$ as in Lemma 4.3.5. Then $\vartheta_N^*[\omega] = 0$.*

Proof. Let $i : \mathrm{SO}(2n) \to \mathrm{SO}(2n+1)$ be the inclusion. The cohomologies of $\mathrm{SO}(2n+1)$ and $\mathrm{SO}(2n)$ are isomorphic up to degree $2n$ and $i^* : H^{2n}(\mathrm{SO}(2n+1)) \to H^{2n}(\mathrm{SO}(2n))$ is an isomorphism. The class $[i^*\vartheta_N^*\omega]$ is the obstruction against a homotopy moment map for the $\mathrm{SO}(2n)$-action on S^{2n}. We know from Lemma 4.3.5, that this action admits a moment map, i.e. $[i^*\vartheta^*\omega] = 0 \in H^{2n}(\mathrm{SO}(2n))$. But as i^* is an isomorphism, this implies that $[\vartheta_N^*\omega] = 0 \in H^{2n}(\mathrm{SO}(2n+1))$. □

Thus, we can use Lemma 4.3.5 to conclude that the action admits a homotopy moment map. Any reader interested in the proof of the general case and/or the explicit construction of the homotopy moment map for the $SO(2n+1)$ action is directed towards section 3 of [MR20].

Conclusion

In this thesis, we have investigated the existence of homotopy moment maps for Lie group actions on multisymplectic manifolds. The guiding theme has been to identify precise conditions for existence and to understand how these conditions reflect and extend the well-established picture in symplectic geometry. By developing several distinct but related approaches, we established precise conditions under which such maps can be constructed.

One approach was through the double complex associated to a Lie group action on a multisymplectic manifold, where the existence of a homotopy moment map was shown to be equivalent to the exactness of the extended multisymplectic form (Theorem $4.1.2$). The cohomology class of this extended multisymplectic form is called the obstruction class for the action of the Lie group. For a compact Lie group, the obstruction class can be seen as an element in the cohomology of the product manifold $G \times M$ (Theorem $4.3.1$) instead of a double complex. Another viewpoint arose from equivariant cohomology and the Cartan model, in which the existence of an equivariant extension of the multisymplectic form ensures the presence of a homotopy moment map (Proposition $4.2.6$).

Using the graded double complex and the Cartan model, we were able to illustrate the existence of homotopy moment maps for several explicit examples, ranging from linear and cotangent-type actions (Examples $4.1.5$ and $4.1.4$ respectively), manifolds where the n-plectic form had an invariant primitive (Example $4.1.3$), to actions of compact Lie groups on spheres equipped with the standard volume form (Proposition $4.3.4$).

The results obtained here therefore not only provide constructive methods for establishing the existence of homotopy moment maps but also clarify the precise sense in which multisymplectic geometry extends the familiar Hamiltonian framework of symplectic geometry.

Appendix A

Equivariant Cohomology

In Section 4.2, we established a key result using the Cartan model for equivariant cohomology. In this appendix, we briefly review the notion of equivariant cohomology and present a result that illustrates its connection to the existence of homotopy moment maps without relying on any specific model. Our discussion is primarily based on Section 4.2 of [GS99] and Section 6.3 of [CFRZ16].

Definition A.0.1. Let $\vartheta : G \times M \to M$ be the action of a compact Lie group G on a manifold M. Let EG be a contractible space on which G acts freely by ϑ^{EG}. Then, the **Equivariant Cohomology** of M is defined as

$$H^\bullet_G(M) := H^\bullet((M \times EG)/G)$$

where G acts on $M \times EG$ diagonally.

Remark A.0.2. The equivariant cohomology of a Lie group action on a manifold is always defined. If the action of G on M is smooth, meaning the projection map is a surjective submersion and M/G is a well-defined manifold, then

$$H^\bullet_G(M) = H^\bullet_{dR}(M/G).$$

To illustrate how equivariant cohomology of a multisymplectic manifold naturally relates to the existence of homotopy moment maps, we consider a compact Lie group action:
$\vartheta : G \times M \to M$. Then, we have

$$G \times (M \times EG) \underset{\pi}{\overset{\vartheta \times \vartheta^{EG}}{\rightrightarrows}} M \times EG \xrightarrow{\quad q \quad} (M \times EG)/G$$

Here, q is a projection to orbits. This induces the following sequence on cohomology:

$$H^\bullet(G \times M) \xleftarrow{\vartheta^* - \pi^*} H^\bullet(M) \xleftarrow{\quad q^* \quad} H^\bullet_G(M)$$

Here, we use the fact that EG is contractible. Now, by definition, we can see that $q \circ \vartheta = q \circ \pi$, so we have that

$$(\vartheta^* - \pi^*) \circ q^* = 0$$

Using Corollary 4.3.3, we can deduce the following:

Theorem A.0.3 (Theorem 1.30 in [MR20])**.** *Let (M, ω) be a multisymplectic manifold and $\vartheta : G \times M \to M$ be a compact Lie group action preserving ω. Then the action admits a homotopy moment map if $[\omega]$ lies in the image of $q^* : H^\bullet_G(M) \to H^\bullet(M)$.*

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